Notes on Belief, Rationality and PBE

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1 Introduction

In the previous quarter of studying game theory, I found the process quite messy, with no general steps for finding the Perfect Bayesian Nash Equilibrium (PBE) in a specific game. In these notes, I aim to provide a systematic approach to finding the PBE.

2 Belief and Sequentially Rational

Definition 1 *A system of beliefs μ is a specification of a probability $\mu(x) \in [0,1]$ for each decision node x such that*

$$\sum_{x \in H} \mu(x) = 1$$

for al information sets H.

Here I will quote Mas-Colell et al. (1995)'s explanation to this: A system of beliefs can be thought of as specifying, for each information set, a probabilistic assessment by the player who moves at that set of the relative likelihoods of being at each of the information set's various decision nodes, conditional upon play having reached that information set.

Let $\mathbb{E}[u_i|H, \mu, \sigma_i, \sigma_{-i}]$ be the player i's expected utility starting at her information set H if her beliefs regarding the conditional probabilities of being at the various nodes in H are given by μ, if she follows strategy σ_i, and if her rivals use strategies σ_{-i}.

Then the definition of sequentially rational is straight forward:

Definition 2 *A strategy profile $\sigma = (\sigma_1, \cdots, \sigma_N)$ is sequentially rational at information set H given a system of beliefs μ if, suppose $\iota_(H)$ is the player who moves at information set H, we have*

$$\mathbb{E}[u_{\iota(H)}|H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq \mathbb{E}[u_{\iota(H)}|H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile σ satisfies this conditional for all information sets H, then we say that σ is sequentially rational given belief system μ.

Sequentially rational is not enough. Whenever possible, beliefs must be consistent with the strategies. To motivate, consider a special case: each player's equilibrium strategy assigns a strictly positive probability to each possible action at every one of her information sets[1]. In this case, the player would assign conditional probabilities of being at each of these nodes given that play has reached this information set using *Bayes' rule*.

$$\mathbb{P}(x|H,\sigma) = \frac{\mathbb{P}(x|\sigma)}{\sum_{x' \in H} \mathbb{P}(x'|\sigma)}$$

Definition 3 *A profile of strategies and system of beliefs (σ, μ) is a weak perfect Bayesian equilibrium (weak PBE) if it has the following properties:*

1. *The strategy profile σ is sequentially rational given belief system μ.*

2. *The system of beliefs μ is derived from strategy profile through Bayes' rule whenever possible. That is, for any information set H such that $\mathbb{P}(H|\sigma) > 0$, we must have*

$$\mu(x) = \frac{\mathbb{P}(x|\sigma)}{\mathbb{P}(H|\sigma)}, \quad \forall x \in H$$

where $\mathbb{P}(H|\sigma) = \sum_{x \in H} \mathbb{P}(x|\sigma)$ is the probability of reaching information set H under strategies σ.

Example 9.C.3 in Mas-Colell et al. (1995) is a very good example to illustrate this concept.

Definition 4 *A strategy profile and system of beliefs (σ, μ) is a sequential equilibrium if it has the following properties:*

(i) Strategy profile σ is sequentially rational given belief system μ.

(ii) There exists a sequence of completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$, with $\lim_{k\to\infty} \sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty} \mu^k$, where μ^k denotes the beliefs derived from strategy profile σ^k using Bayes' rule.

From the definition we can see that a sequential equilibrium must be a weak Perfect Bayesian equilibrium but the converse, in general is not true.

Also, every sequential equilibrium is an SPNE.

3 Signaling Games and Forward Induction

In using forward induction, a player reasons about what could have rationally happened perviously.

Now we consider general signaling games:

- There are two players: Sender (player 1) and Receiver (player 2)

[1]This is called a completely mixed strategy

- Nature picks $t \in T$ type of sender with probability $p(t)$.
- Sender observes t, selects signal $s \in S$
- Receiver observes s (but not t), selects action $a \in A$
- Each player has payoff $U_i(a, t, s)$

In such a game, consider sender's strategy, $\sigma : T \to \Delta(S)$ and receiver's strategy $\alpha : S \to \Delta(A)$. Also, receiver's belief (assessment) is given by $\mu(\cdot|s) \in \Delta(T), \forall s \in S$.

(σ, α, μ) is a perfect Bayesian equilibrium if:

(i) $\sigma(t) \in \Delta(S)$ solves

$$\max_{s \in S} \quad U_1(\alpha(s), t, s)$$

for all t.

(ii) $\alpha(s)$ solves

$$\max_{a \in A} \quad \sum_{t \in T} \mu(t|s) U_2(a, t, s)$$

for all s

(iii) μ derives from prior and σ using Bayes' rule whenever possible

$$\mu(t|s) = \frac{p(t)\sigma(t)(s)}{\sum_{t' \in T} p(t')\sigma(t')(s)}$$

This formula can be understood as

$$\mathbb{P}(t|s) = \frac{\mathbb{P}(s|t)\mathbb{P}(t)}{\sum_{t' \in T} \mathbb{P}(s|t')\mathbb{P}(t')}$$

Also, there are some terminologies: we say that s is on the equilibrium path if $\sigma(t)(s) > 0$ for some t. The equilibrium outcome is $\pi \in \Delta(T \times S \times A)$. It is induced by some PBE.

Definition 5 *Pooling equilibrium: if $\sigma(t)$ is constant in t, i.e. all types of players play the same strategy. $\mu(\cdot|s) = p(\cdot)$ whenever s is on the equilibrium path.*

Separating equilibrium: the support of $\sigma(t)$ does not intersect the support of $\sigma(t')$ if $t \neq t'$. That is, every type of player takes a different action. $\mu(t|s) = 1$ if $\sigma(t)(s) > 0$.

A "no strictly dominated strategy" refinement of PBE: for every off-equilibrium signal s, receiver's belief $\mu(\cdot|s)$ places probability 0 on strictly dominated strategies whenever possible. The strictly dominance is defined in the following way: s is strictly dominated if $\exists s' \in S$ such that

$$\min_{a \in A} U_1(a, t, s') > \max_{a \in A} U_1(a, t, s)$$

References

Mas-Colell, A., M. D. Whinston, J. R. Green, et al. (1995). *Microeconomic theory*, Volume 1. Oxford university press New York.