Arbitrage-Free Factor Analysis: An Application to Affine Term-Structure Models

Jun Liu[1], Jun Pan[2] and Jiahui Shui[1]

[1]Rady School of Management, UCSD
[2]SAIF, Shanghai Jiao Tong University

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Motivation

- Dynamic asset pricing models often employ latent state variables to model fundamental risk factors
- A small number of latent state variables connect a broad set of cross-sectional asset prices via no-arbitrage restrictions
- Conversely, the market-observed cross-sectional prices can be used to extract risk factors governed by the state variables under no-arbitrage conditions
- However, estimating those models is challenging (e.g. QMLE)

Arbitrage-Free Cross-Sectional Factors

- Assuming the existence of a risk-neutral measure, the excess return of asset i

$$\frac{\mathrm{d}P_{it}}{P_{it}} - r_t \mathrm{d}t = b_i \mathrm{d}F_t^Q + \mathrm{d}e_{it}, \quad i = 1, 2, \cdots, N \tag{1}$$

- The mean of the latent risk factors, $\mathrm{d}F_t^Q$, is zero under the risk-neutral measure $\mathbb{Q}$ and non-zero under physical measure $\mathbb{P}$.
- $\mathrm{d}e_t$ is zero mean under both $\mathbb{Q}$ and $\mathbb{P}$, representing pricing errors.
- Under physical measure $\mathbb{P}$:

$$\mathrm{d}F_t^Q = \Lambda_t \mathrm{d}t + \mathrm{d}F_t \tag{2}$$

where Λ_t is the vector of risk premiums corresonding to each risk factor.

Motivation

- Fama and French (2020): Exogenously specified factor loadings, such as values of size, book-to-market ratio, etc
- We use model-implied factor loadings to extract the cross-section factors.
- Consequence: corresponding cross-section factors are $\mathrm{d}F_t^Q$ are the risk-neutral cross-section factors
- By contrast, the cross-section factors from Fama and French (2020) are long-short portfolios associated with the characteristics

Affine Term Structure Model

- Canonical form: The instantaneous nominal interest rate is affine in state variables:

$$r_t = \delta_0 + \mathbf{1}' x_t \tag{3}$$

- Dynamics of x_t under risk neutral measure $\mathbb{Q}$:

$$\mathrm{d}x_t = -K^Q x_t \mathrm{d}t + \sigma \mathrm{d}B_t^Q \tag{4}$$

- The zero-coupon bond price at time t with maturity $\tau = T - t$ is given by

$$p_t = \mathbb{E}\left[e^{-\int_t^T r_s \mathrm{d}s}\right] = e^{a(\tau)+b(\tau)' x_t} \tag{5}$$

- By Itô's lemma, we have

$$\frac{\mathrm{d}p_t}{p_t} = r_t \mathrm{d}t + b(\tau)' \sigma \mathrm{d}B_t^Q \tag{6}$$

Pricing Equation

- Suppose we observed that

$$\frac{\mathrm{d}P_t}{P_t} = r_t \mathrm{d}t + b' \sigma \mathrm{d}B_t^Q + \sigma_e \mathrm{d}B_t^e \tag{7}$$

  where $\sigma_e \mathrm{d}B_t^e$ are pricing errors.
- Here, our estimated factors are $\sigma \mathrm{d}B_t^Q$
- Moreover, suppose the market price of risk has the following form

$$\Lambda_t = \lambda_0 + \lambda_1 x_t \tag{8}$$

- We can further estimate λ_0 and λ_1 via time-series regression.

Estimation

- Factors are obtained from regression:

$$\sigma \mathrm{d}B_t^Q = \Delta b(\Delta b' \Delta b)^{-1} \Delta b \Delta P \tag{9}$$

  where Δb and ΔP are corresponding demeaned processes.
- Minimize the following problem:

$$\min_{K^Q} \quad f = \sum_{t=1}^{T} \left(\Delta P_t - \Delta b \sigma \mathrm{d}B_t^Q \right)' (\Delta P_t - \Delta b \sigma \mathrm{d}B_t^Q) \tag{10}$$

- After getting K^Q, generate x_t. Guess x_0
- Using regression to get λ_0 and λ_1 from

$$\sigma \mathrm{d}B_t^Q = \sigma \lambda_1 x_t \mathrm{d}t + \sigma \lambda_0 \mathrm{d}t + \sigma \mathrm{d}B_t \tag{11}$$

Simulations

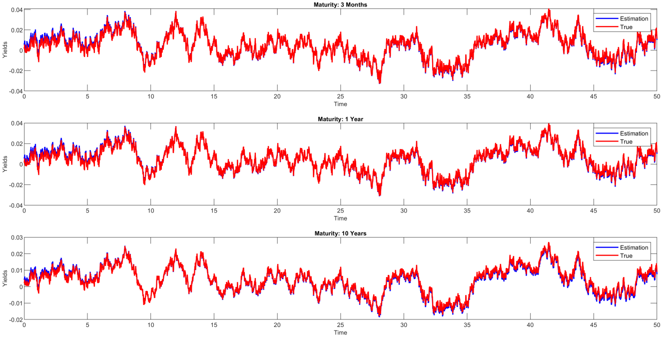

Figure: Comparison between estimated yield and simulated yield

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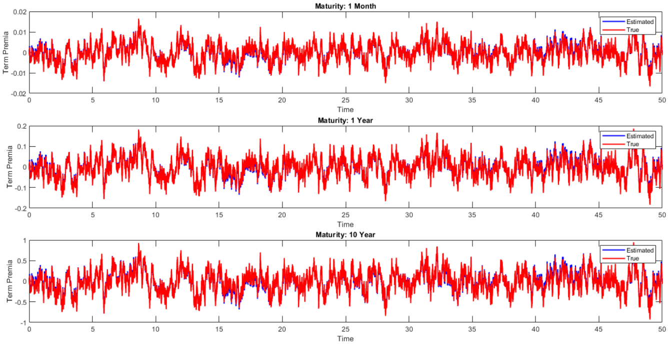

Figure: Comparison between estimated market price of risk and simulated market price of risk

Preliminary Result

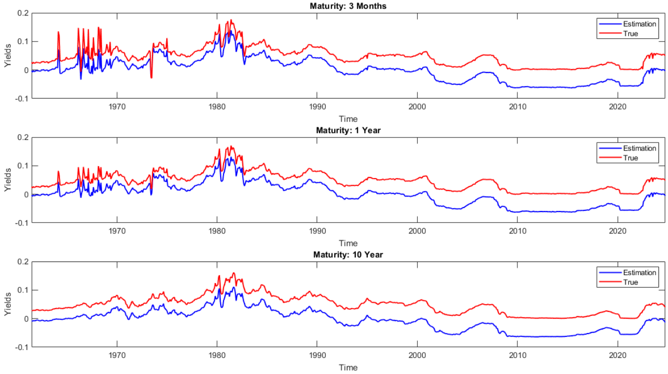

Figure: Fitted Yield

Preliminary Result

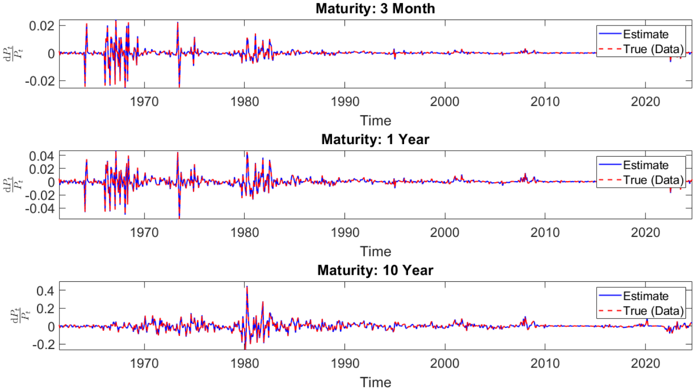

Figure: Fitted Bond Return. Investor care returns rather than yields!

Possible Improvements

- Estimating x_0 from data
- Deviate from no arbitrage assumption
- State-dependent b
- Bond trading data rather than 'constructed' yield data