Affine Term Structure Model

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Notations

- Let $P(t,T)$ be the price of a zero-coupon bond that pays \$1 at maturity T and $p(t,T) = \ln P(t,T)$ be the log price.
- The bond price is given by

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s}\right] = \mathbb{E}_t\left[\frac{\xi_T}{\xi_t}\right]$$

  where ξ_t is the SDF process:

$$\xi_t = \exp\left\{-\frac{1}{2}\int_0^t \Lambda_s^\top \Lambda_s \mathrm{d}s - \int_0^t \Lambda_s \mathrm{d}B_s\right\}$$

- The yield is given by

$$y(t,T) = -\frac{1}{T-t}\ln P(t,T) = -\frac{1}{T-t}p(t,T)$$

Bond Pricing

- The pricing relation shows that the term structure model consists two parts:
  - Change of measure: From $\mathbb{P}$ to $\mathbb{Q}$.
  - The dynamics of short rate under $\mathbb{Q}$. In factor models, this can be written as $r_t = h(x_t)$, where x_t is a time-homogeneous Markov process under $\mathbb{Q}$.
- Then the bond price is also a function of x_t (and maturity $\tau := T - t$), $P(t, T) = H(x_t, \tau)$
- Therefore, yield is also a function of x_t.
- What if h has a linear functional form? $\Rightarrow$ *Affine term structure models*.

Affine Term Structure Model

- Under $\mathbb{Q}$, the dynamics of x_t are given by

$$\mathrm{d}x_t = -K^{\mathbb{Q}} x_t \mathrm{d}t + \sigma \mathrm{d}B_t^{\mathbb{Q}}$$

- The short rate is *affine* in factors: $r_t = \delta_0 + \delta_1^\top x_t$
- Then Duffie and Kan (1996) proved that
  - The bond price is (exponentially) *affine* in factors: $P(t,T) = e^{a(\tau)+b(\tau)^\top x_t}$
  - The yield is also*affine* in factors: $y(t,T) = A(\tau) + B(\tau)^\top x_t$
- What about dynamics under $\mathbb{P}$? If $\Lambda_t = \lambda_0 + \lambda_1 x_t$, which is *affine* in factors, then the dynamics of x_t under $\mathbb{P}$ can be represented as

$$\mathrm{d}x_t = -K^{\mathbb{P}}(x_t - \bar{x}^{\mathbb{P}})\mathrm{d}t + \sigma \mathrm{d}B_t^{\mathbb{P}}$$

Discrete-Time Analogy

- Let $P_t^{(n)}$ denote the bond price at time t with maturity n. $p_t^{(n)} := \ln P_t^{(n)}$
- The state variables follow (under $\mathbb{P}$)

$$X_{t+1} = \mu + \Phi X_t + v_{t+1}, \quad v_{t+1} \sim N(0, \Sigma)$$

  Under $\mathbb{Q}$:

$$X_{t+1} = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_t + v_{t+1}^{\mathbb{Q}}$$

- The short rate is still affine in X_t: $r_t = \delta_0 + \delta_1^\top X_t$.
- The pricing kernel is given by

$$M_{t+1} = \exp\left\{-r_t - \frac{1}{2}\Lambda_t^\top \Sigma \Lambda_t - \Lambda_t^\top v_{t+1}\right\}$$

- Log bond price is affine in factors:

$$p_t^{(n)} = a_n + b_n^\top X_t$$

Estimation Methods

- MLE: Ait-Sahalia and Kimmel (2010), Joslin et al. (2011)
- GMM: Dai and Singleton (2000) (Simulation-based Method of Moments)
- Hamilton and Wu (2012): Minimize chi square statistic for test that restrictions are valid. Asymptotically equivalent to MLE, but simpler.
- **OLS**: Adrian et al. (2013).

OLS Estimation: ACM

- Log excess return: $rx_{t+1}^{(n-1)} := p_{t+1}^{(n-1)} - p_t^{(n)} - r_t$
- $r_t = -\ln P_t^{(1)}$ is the continuously compounded risk-free rate.
- Suppose that the pricing error is $e_{t+1}^{(n-1)}$, which is orthogonal to v_{t+1}.
- From the model, we can derive that ▸ Bond Excess Return Derivation

$$rx_{t+1}^{(n-1)} = \underbrace{\beta^{(n-1)\top}\Sigma^{1/2}(\lambda_0 + \lambda_1 X_t)}_{\text{Expected return}} - \underbrace{\frac{1}{2}\left(\beta^{(n-1)\top}\Sigma\beta^{(n-1)} + \sigma^2\right)}_{\text{Convexity adjustment}}$$

$$+ \underbrace{\beta^{(n-1)\top} v_{t+1}}_{\text{Priced return innovation}} + \underbrace{e_{t+1}^{(n-1)}}_{\text{Return Pricing Error}}$$

ACM Estimation

- Observed factors (First K PCs of yield curve): X_t. Use OLS (VAR) to get $\hat{\Phi}, \hat{\mu}$ and $\hat{v}_{t+1}$.
- Run Regression: $rx_{t+1}^{(n-1)}$ on $Z = [1, \hat{v}_{t+1}, X_t]$:

$$[\hat{\alpha}, \hat{\beta}, \hat{\gamma}] = rxZ^\top (ZZ^\top)^{-1}$$

- Note that according to model

$$\alpha = \beta^\top \lambda_0 - \frac{1}{2}(B^* \mathrm{vec}(\hat{\Sigma}) + \sigma^2 \iota_N)$$

  and $\gamma = \beta' \lambda_1$
- Thus

$$\hat{\lambda}_0 = (\hat{\beta}\hat{\beta}^\top)^{-1}\hat{\beta}(\hat{\alpha} + \frac{1}{2}(B^* \mathrm{vec}(\hat{\Sigma}) + \sigma^2 \iota_N))$$
$$\hat{\lambda}_1 = (\hat{\beta}\hat{\beta}^\top)^{-1}\hat{\beta}\hat{\gamma}$$

Affine Yields

- Let $u_t^{(n)}$ denote the pricing error on the log prices: $p_t^{(n)} = a_n + b_n^\top X_t + u_t^{(n)}$
- Also note that $rx_{t+1}^{(n-1)} = p_{t+1}^{(n-1)} - p_t^{(n)} + p_t^{(1)}$
- System of linear restrictions:

$$a_n = a_{n-1} + b'_{n-1}(\mu - \Sigma^{1/2}\lambda_0) + \frac{1}{2}(b'_{n-1}\Sigma b_{n-1} + \sigma^2) - \delta_0$$

$$b_n = b'_{n-1}(\Phi - \Sigma^{1/2}\lambda_1) - \delta'_1$$

$$a_0 = 0, \quad b'_0 = 0, \quad \beta^{(n)} = b'_n$$

- Also, the log bond pricing errors can be decomposed as

$$\underbrace{u_{t+1}^{(n-1)} - u_t^{(n)} + u_t^{(1)}}_{\text{Log yield pricing error}} = \underbrace{e_{t+1}^{(n-1)}}_{\text{return pricing error}}$$

- "Risk-neutral Yields": Set $\lambda_0 = 0, \lambda_1 = 0$ and then obtain a_n^*, b_n^*. Define $y_{rn} = -\frac{1}{n}(a_n^* + b_n^* x_t)$: yields when $\mathbb{P} = \mathbb{Q}$.

A Three-factor Example - PCA

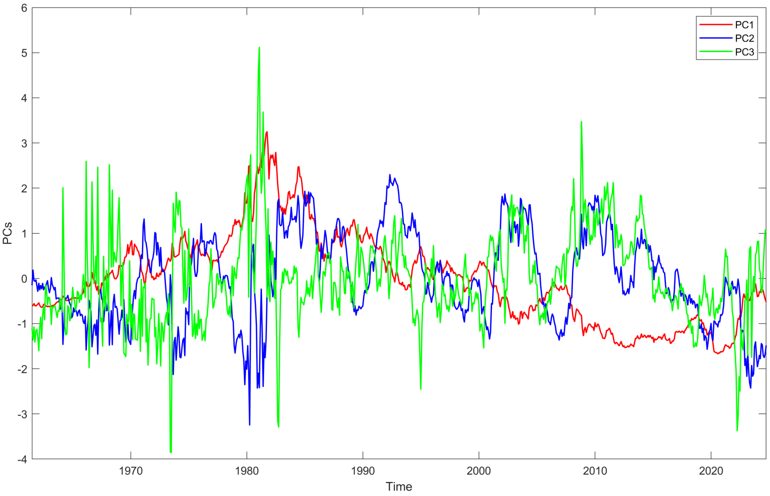


Figure: First three PCs (normalized)

A Three-factor Example

- Data: Gürkaynak et al. (2007). Sample Period: From 1961-06 to 2024-09
- Consider the first three (Normalized) PCs as state variables, i.e. mean 0 and std 1.
- Step 1: Estimate VAR(1):

$$X_{t+1} = \mu + \Phi X_t + v_{t+1}$$

- Estimated $\hat{\Phi}$

$$\begin{pmatrix} 0.9938 & -0.0041 & -0.0043 \\ 0.0110 & 0.9523 & 0.0764 \\ 0.0087 & 0.0163 & 0.8004 \end{pmatrix}$$

- PC1: highly persistent

A Three-factor Example

- Step 2: Run regression $rx_{t+1}^{(n-1)}$ on $Z = [1, \hat{v}_{t+1}, X_t]$:

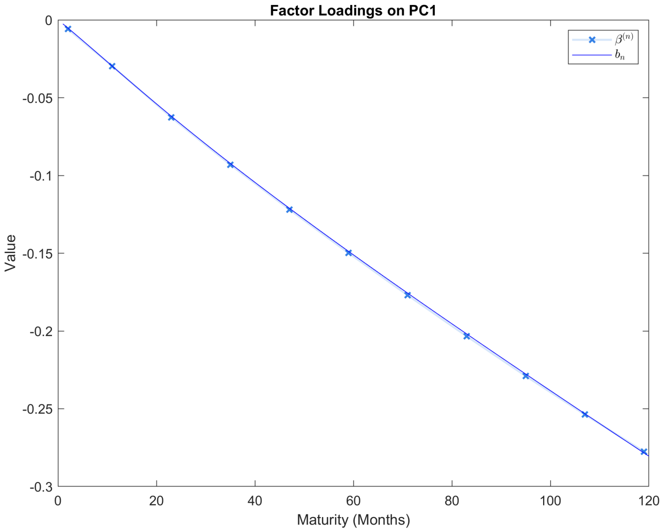
$$[\hat{\alpha}, \hat{\beta}, \hat{\gamma}] = rxZ'(ZZ')^{-1}$$

  get $\hat{\lambda}_0$ and $\hat{\lambda}_1$
- Step 3: Run regression

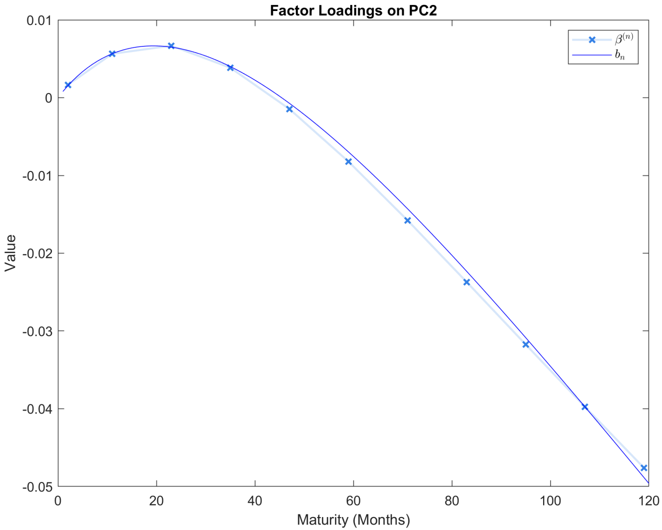
$$r_t = \delta_0 + \delta_1' x_t$$

- Step 4: Use $\hat{\delta}_0$ and $\hat{\delta}_1$ with system of linear restrictions to get a_n and b_n.
- **Question**: Is this b_n the same as the $\beta^{(n)}$ obtained from regression? *There are gaps.*

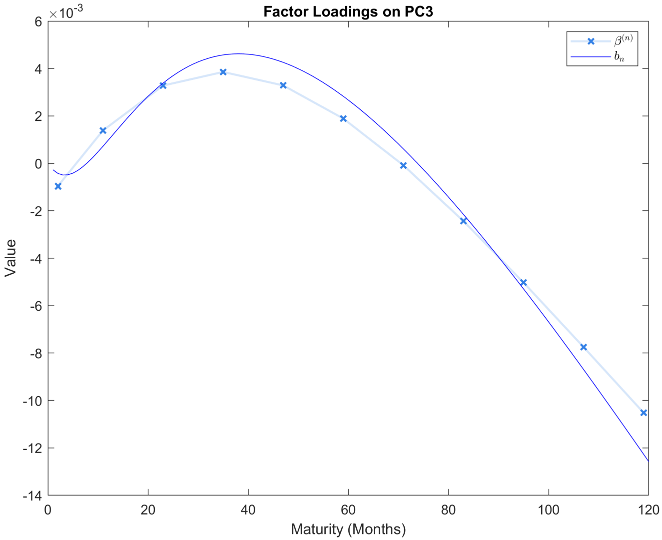
Factor Loadings



Factor Loadings



Factor Loadings



Spanning Hypothesis

- Yields$= f(X_t)$ is a function of state variables. ?$\Rightarrow X_t = f^{-1}$(Yields)
- Yields must contain information about evolution in underlying state variables
- The question is, do yields contain **all** the information about underlying risk factors?
- In other words, is there any variables can forecast future yields and return *given current yields*?
- **Yes**: Ludvigson and Ng (2009), Joslin et al. (2014), Huang and Shi (2023), ...
- Probably **No**: Bauer and Hamilton (2018), Duffee (2013), ...

ACM Extensions

- Now consider spanning factors (from yields) X_t^s and unspanning factors X_t^u.
- The spanning restriction is that the risk exposures of the unspanned factors are equal to zero, i.e. $\beta^{(n)} = [\beta_s^{(n)}, 0]$
- Fitting results show that the unspanned macro factor specification provides a somewhat poorer fit to cross section of Treasury yields than four- and five- factor specifications.

Term Premium

- Term Premium is defined as difference between yields and risk-neutral yields.
- Updated daily at Federal Reserve Bank of New York
- I will introduce more about term premium in next slides.

Back

Bond Excess Return Derivation

- For continuous-time model: By Itô's lemma, the log bond excess return is given by

$$\mathrm{d}\ln P(t,T) - r_t \mathrm{d}t = b\sigma \mathrm{d}B_t^{\mathbb{Q}}$$

- For discrete-time model: note that

$$P_t^{(n)} = \mathbb{E}_t[M_{t+1}P_{t+1}^{(n-1)}] \Rightarrow 1 = \mathbb{E}\left[\exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\Lambda_t^\top \Sigma \Lambda_t - \Lambda_t v_{t+1}\right)\right]$$

- Note that $rx_{t+1}^{(n-1)}$ and v_{t+1} are joint normally distributed., then

$$\mathbb{E}_t[rx_{t+1}^{(n-1)}] = \mathrm{Cov}_t[rx_{t+1}^{(n-1)}, v_{t+1}^\top \Lambda_t] - \frac{1}{2}\mathrm{Var}_t[rx_{r+1}^{(n-1)}]$$

- Denote

$$\beta_t^{(n-1)} = \Sigma^{-1}\mathrm{Cov}_t(rx_{t+1}^{(n-1)}, v_{t+1}^\top)$$

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