# Credit Yield Spread Propositions

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# Yield Spread

- Definition:

$$e^{-\Delta_0 T} = \frac{D_0}{FP(0, T)}$$

- Introduce $T$-forward measure $\mathbb{Q}^T$:

$$\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \frac{e^{-\int_0^T r_s \mathrm{d}s}}{\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_0^T r_s \mathrm{d}s}]}$$

- Alternative expression:

$$e^{-\Delta_0 T} = \frac{1}{F}\mathbb{E}^{\mathbb{Q}^T}[\min(V_T, F)]$$

# Heston Model

- Firm value

$$\mathrm{d}V_t = V_t(r_t\mathrm{d}t + \sigma_1\sqrt{v_t}\mathrm{d}B_t^{\mathbb{Q}} + \sigma_2\mathrm{d}Z_t^{\mathbb{Q}})$$

- Short rate

$$\mathrm{d}r_t = -K^{\mathbb{Q}}(r_t - \bar{r}^{\mathbb{Q}})\mathrm{d}t + \sigma_r\sqrt{v_t}\mathrm{d}B_{rt}^{\mathbb{Q}}$$

- Volatility process

$$\mathrm{d}v_t = -K(v_t - \bar{v}^{\mathbb{Q}})\mathrm{d}t + \sigma_v\sqrt{v_t}\mathrm{d}B_{vt}^{\mathbb{Q}}$$

- Correlation: $\mathrm{d}B_t^{\mathbb{Q}}\mathrm{d}B_{rt}^{\mathbb{Q}} = \rho_r\mathrm{d}t$, $\mathrm{d}B_t^{\mathbb{Q}}\mathrm{d}B_{vt}^{\mathbb{Q}} = \rho_v\mathrm{d}t$, $\mathrm{d}B_{rt}^{\mathbb{Q}}\mathrm{d}B_{vt}^{\mathbb{Q}} = \rho_{rv}\mathrm{d}t$. $Z_t^{\mathbb{Q}}$ is independent of other Brownian motions.

# $T$-forward Measure

- Under $T$-forward measure, the log firm value process $(x_t)$ has following dynamic:

$$\mathrm{d}x_t = \left( r_t + \left( b(t)\rho_r\sigma_1\sigma_r + \rho_v h(t)\sigma_1\sigma_v - \frac{1}{2}\sigma_1^2 \right) v_t - \frac{1}{2}\sigma_2^2 \right) \mathrm{d}t + \sigma_1\sqrt{v_t}\mathrm{d}B_t^{\mathbb{Q}^T} + \sigma_2\mathrm{d}Z_t^{\mathbb{Q}^T}$$

  where $b(t)$ and $h(t)$ are functions in the zero-coupon bond price:

$$P(t, T) = e^{a(t,T)+b(t,T)r+h(t,T)v}$$

- Short rate:

$$\mathrm{d}r_t = [-K^{\mathbb{Q}}(r_t - \bar{r}^{\mathbb{Q}}) + (b(t)\sigma_r^2 + \rho_{rv}h(t)\sigma_r\sigma_v)v_t]\mathrm{d}t + \sigma_r\sqrt{v_t}\mathrm{d}B_{rt}^{\mathbb{Q}^T}$$

# Monotonicity of Short Rate

- Consider two short rate process: $r_t^{\mathcal{A}}$ and $r_t^{\mathcal{B}}$ with $r_0^{\mathcal{A}} > r_0^{\mathcal{B}}$.
- Claim: $r_t^{\mathcal{A}} > r_t^{\mathcal{B}}$ for all $t \in [0, T]$.
- Sketch of proof: If there exists $t_0 \in (0, T]$ such that $r_{t_0}^{\mathcal{A}} = r_{t_0}^{\mathcal{B}}$, then it $r_t^{\mathcal{A}}$ will be identical to $r_t^{\mathcal{B}}$ onward. By countinuity argument, this contradicts to $r_0^{\mathcal{A}} > r_0^{\mathcal{B}}$.

# Firm Value under $\mathbb{Q}^T$

- Firm value

$$V_T = V_0 \exp\left[\int_0^T \left(r_t + \left(b(t)\rho_r\sigma_1\sigma_r + \rho_v h(t)\sigma_1\sigma_v - \frac{1}{2}\sigma_1^2\right) v_t \right.\right.$$
$$\left.\left. -\frac{1}{2}\sigma_2^2\right) \mathrm{d}t + \sigma_1 \int_0^T \sqrt{v_t}\mathrm{d}B_t^{\mathbb{Q}^T} + \sigma_2 Z_T^{\mathbb{Q}^T}\right]$$

- $V_T$ is strictly **increasing** with $r_0$ under $\mathbb{Q}^T$.
- The yield spread is **decreasing** with $r_0$ since

$$\Delta_0 = -\frac{1}{T}\ln\frac{1}{F}\mathbb{E}^{\mathbb{Q}^T}[\min(V_T, F)]$$

# CIR Model

- Firm Value:

$$\mathrm{d}V_t = r_t V_t \mathrm{d}t + \sigma_1 \sqrt{r_t} V_t \mathrm{d}B_{rt}^Q + \sigma_2 V_t \mathrm{d}B_t^Q$$

- Short Rate:

$$\mathrm{d}r_t = -K^Q(r_t - \bar{r}^Q)\mathrm{d}t + \sigma_r \sqrt{r_t} \mathrm{d}B_{rt}^Q$$

- Correlation: $\mathrm{d}B_t^Q \mathrm{d}B_{rt}^Q = \rho \mathrm{d}t$

# $T$-forward Measure

- Under $\mathbb{Q}^T$, log firm value $(x_t)$ follows:

$$\mathrm{d}x_t = \left[ r_t + b(t,T)\sigma_1\sigma_r r_t - \frac{1}{2}(\sigma_1^2 r_t + \sigma_2^2) \right] \mathrm{d}t + \sigma_1\sqrt{r_t}\mathrm{d}B_{rt}^{\mathbb{Q}^T} + \sigma_2 \mathrm{d}B_t^{\mathbb{Q}^T}$$

  where $b(t,T)$ is given in the zero-coupon price:

$$P(t,T) = e^{a(t,T)+b(t,T)r_t}$$

- Short Rate:

$$\mathrm{d}r_t = [-K^{\mathbb{Q}}(r_t - \bar{r}^{\mathbb{Q}}) + b(t,T)\sigma_r^2 r_t]\mathrm{d}t + \sigma_r\sqrt{r_t}\mathrm{d}B_{rt}^{\mathbb{Q}^T}$$

# Coefficient of $r_t$

- CIR model has been proved that has unique strong solution[1].
- The property $r_t^{\mathcal{A}} > r_t^{\mathcal{B}}$ when $r_0^{\mathcal{A}} > r_0^{\mathcal{B}}$ preserves.
- The coefficient of $r_t$ in $x_t$ is

$$1 + b(t, T)\sigma_1\sigma_r - \frac{1}{2}\sigma_1^2$$

- When $\sigma_1$ is large enough, $r_0$ will negatively contribute to $x_t$ hence the yield spread will increase
- When $\sigma_1 = 0$, the yield spread is decreasing with $r_0$.

[1] Yamada and Watanabe (1971)

# Conclusion

- To establish an increasing relationship between $r_0$ and $\Delta_0$, we must incorporate $r_t$ into the volatility term of $V_t$.