

# Review Session 3: Volatility Modeling

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# Introduction to Volatility Modeling

- ▶ Realized Volatility: Note that

$$\text{Var}_t(r_{t+1}) = \mathbb{E}_t[(r_{t+1} - \mathbb{E}_t(r_{t+1}))^2] = \mathbb{E}_t[r_{t+1}^2] - (\mathbb{E}_t[r_{t+1}])^2$$

- ▶ One way to estimate is using residual:  $e_t = r_{t+1} - \mathbb{E}_t[r_{t+1}]$ , suppose that  $i = 1, \dots, n$  are daily residual, then an estimation for monthly volatility is

$$\hat{\sigma}_t^2 = \left( \sum_{i=1}^n e_i^2 \right) \quad (1)$$

# Introduction to Volatility Modeling

- ▶ Implied Volatility: From option prices.
- ▶ Recall Black-Scholes Formula:

$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2) \quad (2)$$

where

$$d_1 = \frac{\ln \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad (3)$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

- ▶ If we know  $S_0, K, r, T$ , then we can solve the  $\sigma$ . Volatility obtained from options prices are so-called implied volatilities.

# ARCH Model

- ▶ The return process

$$r_t = \mu + \phi r_{t-1} + u_t \quad (4)$$

- ▶ Conditional Mean:  $\mathbb{E}_{t-1}[r_t] = \mu + \phi r_{t-1}$
- ▶ What is unconditional mean and unconditional variance?
- ▶ The innovation:

$$u_t = r_t - \mathbb{E}_{t-1}[r_t]$$

- ▶ Can we have time-varying second moments of  $u_t$ ?
- ▶ Suppose that  $u_t$  follows AR( $p$ ) process:

$$u_t^2 = \zeta + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_p u_{t-p}^2 + w_t \quad (5)$$

- ▶ Although the unconditional mean is constant, conditional mean  $\mathbb{E}_{t-1}[u_t^2]$  is time-varying.
- ▶ Can you use OLS to estimate ARCH? **Yes.**

## GARCH Model

- ▶ One can present ARCH(1) as  $u_t = \sqrt{h_t}v_t$  where  $v_t$  is i.i.d with mean 0 and variance 1.
- ▶ If  $h_t = \zeta + \alpha u_{t-1}^2$ , then it is ARCH(1) model
- ▶ GARCH(1,1):

$$h_t = \zeta + \delta h_{t-1} + \alpha u_{t-1}^2 \quad (6)$$

- ▶  $\delta + \alpha \leq 1$
- ▶ GARCH(p,q):

$$h_t = \zeta + \sum_{i=1}^q \alpha_i u_{t-i}^2 + \sum_{j=1}^p \beta_j h_{t-j}$$

- ▶ Can you use OLS to estimate GARCH? **No.**
- ▶ IGARCH(1,1) is GARCH(1,1) when  $\delta + \alpha = 1$
- ▶ AsyGARCH: Negative surprises increase volatility more than positive surprises

## (G)ARCH Question

Consider the following ARCH(1) model:

$$u_t^2 = \zeta + \alpha u_{t-1}^2 + w_t \quad (7)$$

where  $\text{Cov}(u_{t-1}^2, w_t) = 0$  and  $\alpha < 1$ .

(a) Find  $\mathbb{E}_{t-1}[u_t^2]$  and  $\mathbb{E}[u_t^2]$

## (G)ARCH Question

(b) Now suppose that  $u_t = \sqrt{h_t}v_t$ , where  $v_t$  is i.i.d with mean zero and variance 1. And  $h_t = \zeta + \alpha u_{t-1}^2$ . Find  $\mathbb{E}_{t-1}[u_t^2]$  and  $\mathbb{E}[u_t^2]$  in this case.

## (G)ARCH Question

(c) Consider GARCH(1,1) model:

$$h_t = \zeta + \delta h_{t-1} + \alpha u_{t-1}^2$$

Find  $\mathbb{E}_{t-1}[u_t^2]$  and  $\mathbb{E}[u_t^2]$  in this case.



## MLE for (G)ARCH Question

Suppose that we have residuals  $u_t$  from regression

$Y_t = c + \phi Y_{t-1} + u_t$ . Consider

$$u_t = \sqrt{h_t} v_t, \quad h_t = \zeta + \alpha u_{t-1}^2$$

where  $v_t$  is i.i.d  $N(0, 1)$ . Find the log-likelihood function for  $(\zeta, \alpha)$ .

## Test of Homoskedasticity vs ARCH(p)

- ▶ Consider

$$u_t^2 = \zeta + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_p u_{t-p}^2 + w_t \quad (8)$$

- ▶ Engle (1982): Lagrange Multiplier-type test
  - ▶ Regress  $Y_t$  on  $Y_{t-1}$  to get  $\hat{u}_t$
  - ▶ Regress  $\hat{u}_t^2$  on  $\hat{u}_{t-1}^2, \dots, \hat{u}_{t-p}^2$ .
  - ▶ Get  $R^2$  from this regression
  - ▶  $T \cdot R^2 \sim \chi^2(m)$

## Testing Nested GARCH(p,q) Models

- ▶ Likelihood ratio test
- ▶ Estimating GARCH(1,1), get log-likelihood function  $\Lambda(\theta_0)$
- ▶ Estimating GARCH(2,2), get log-likelihood function  $\Lambda(\theta_1)$
- ▶ Note that GARCH(1,1) is a special case of GARCH(2,2) (take  $\alpha_2 = \delta_2 = 0$ )
- ▶ Hence  $\Lambda(\theta_0) < \Lambda(\theta_1)$
- ▶ Likelihood ratio:

$$LR = 2(\Lambda(\theta_1) - \Lambda(\theta_0)) \sim \chi^2(2) \quad (9)$$

# Estimating ARCH(1) Using Python

- ▶ Python package: arch
- ▶ Sample code:

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```
model = arch_model(ret_data, mean='AR', lags=1, p=1,  
                   q=0)  
am = model.fit()
```

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- ▶ Equivalently:

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```
model = arch_model(ret_data, mean='AR', lags=1,  
                   vol='ARCH', p=1)  
am = model.fit()
```

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- ▶ GARCH(1,1): Set  $q = 1$