Review Session 3: Volatility Modeling

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Introduction to Volatility Modeling

- Realized Volatility: Note that

$$\mathrm{Var}_t(r_{t+1}) = \mathbb{E}_t[(r_{t+1} - \mathbb{E}_t(r_{t+1}))^2] = \mathbb{E}_t[r_{t+1}^2] - (\mathbb{E}_t[r_{t+1}])^2$$

- One way to estimate is using residual: $e_t = r_{t+1} - \mathbb{E}_t[r_{t+1}]$, suppose that $i = 1, \cdots, n$ are daily residual, then an estimation for monthly volatility is

$$\hat{\sigma_t}^2 = \left(\sum_{i=1}^{n} e_i^2\right) \tag{1}$$

Introduction to Volatility Modeling

- Implied Volatility: From option prices.
- Recall Black-Scholes Formula:

$$C_0 = S_0 N(d_1) - K e^{-rT} N(d_2) \tag{2}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned} \tag{3}$$

- If we know S_0, K, r, T, then we can solve the σ. Volatility obtained from options prices are so-called implied volatilities.

ARCH Model

- The return process

$$r_t = \mu + \phi r_{t-1} + u_t \tag{4}$$

- Conditional Mean: $\mathbb{E}_{t-1}[r_t] = \mu + \phi r_{t-1}$
- What is unconditional mean and unconditional variance?
- The innovation:

$$u_t = r_t - \mathbb{E}_{t-1}[r_t]$$

- Can we have time-varying second moments of u_t?
- Suppose that u_t follows AR(p) process:

$$u_t^2 = \zeta + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_p u_{t-p}^2 + w_t \tag{5}$$

- Although the unconditional mean is constant, conditional mean $\mathbb{E}_{t-1}[u_t^2]$ is time-varying.
- Can you use OLS to estimate ARCH? **Yes**.

GARCH Model

- One can present ARCH(1) as $u_t = \sqrt{h_t} v_t$ where v_t is i.i.d with mean 0 and variance 1.
- If $h_t = \zeta + \alpha u_{t-1}^2$, then it is ARCH(1) model
- GARCH(1,1):

$$h_t = \zeta + \delta h_{t-1} + \alpha u_{t-1}^2 \tag{6}$$

- $\delta + \alpha \leq 1$
- GARCH(p,q):

$$h_t = \zeta + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}$$

- Can you use OLS to estimate GARCH? **No**.
- IGARCH(1,1) is GARCH(1,1) when $\delta + \alpha = 1$
- AsyGARCH: Negative surprises increase volatility more than positive surprises

(G)ARCH Question

Consider the following ARCH(1) model:

$$u_t^2 = \zeta + \alpha u_{t-1}^2 + w_t \tag{7}$$

where $\mathrm{Cov}(u_{t-1}^2, w_t) = 0$ and $\alpha < 1$.
(a) Find $\mathbb{E}_{t-1}[u_t^2]$ and $\mathbb{E}[u_t^2]$
Solution: $\mathbb{E}_{t-1}[u_t^2] = \zeta + \alpha u_{t-1}^2$. For the unconditional one, take expectation at both hand sides

$$\mathbb{E}[u_t^2] = \zeta + \alpha \mathbb{E}[u_{t-1}^2] = \zeta + \alpha \mathbb{E}[u_{t-1}^2] \tag{8}$$

Hence

$$\mathbb{E}[u_t^2] = \frac{\zeta}{1-\alpha} \tag{9}$$

(G)ARCH Question

(b) Now suppose that $u_t = \sqrt{h_t} v_t$, where v_t is i.i.d with mean zero and variance 1. And $h_t = \zeta + \alpha u_{t-1}^2$. Find $\mathbb{E}_{t-1}[u_t^2]$ and $\mathbb{E}[u_t^2]$ in this case.

Solution: Since v_t are i.i.d, then

$$\mathbb{E}_{t-1}[u_t^2] = \mathbb{E}_{t-1}[h_t v_t^2] = \mathbb{E}_{t-1}[h_t]\mathbb{E}_{t-1}[v_t^2] = \mathbb{E}_{t-1}[h_t] = \zeta + \alpha u_{t-1}^2 \tag{10}$$

And

$$\mathbb{E}[u_t^2] = \mathbb{E}[h_t v_t^2] = \mathbb{E}[h_t] = \zeta + \alpha \mathbb{E}[u_{t-1}^2] \tag{11}$$

Hence

$$\mathbb{E}[u_t^2] = \frac{\zeta}{1-\alpha} \tag{12}$$

(G)ARCH Question

(c) Consider GARCH(1,1) model:

$$h_t = \zeta + \delta h_{t-1} + \alpha u_{t-1}^2$$

Find $\mathbb{E}_{t-1}[u_t^2]$ and $\mathbb{E}[u_t^2]$ in this case.
Solution: Similarly

$$\begin{aligned} \mathbb{E}_{t-1}[u_t^2] &= \mathbb{E}_{t-1}[h_t v_t^2] = \mathbb{E}_{t-1}[h_t]\mathbb{E}_{t-1}[v_t^2] \\ &= \mathbb{E}_{t-1}[h_t] = \zeta + \delta h_{t-1} + \alpha u_{t-1}^2 \end{aligned} \tag{13}$$

And

$$\mathbb{E}[u_t^2] = \mathbb{E}[h_t v_t^2] = \mathbb{E}[h_t] = \zeta + \delta\mathbb{E}[h_{t-1}] + \alpha\mathbb{E}[u_{t-1}^2] \tag{14}$$

Note that here $\mathbb{E}[h_{t-1}] = \mathbb{E}[u_{t-1}^2]$, hence

$$\mathbb{E}[u_t^2] = \frac{\zeta}{1 - \delta - \alpha} \tag{15}$$

MLE for (G)ARCH Question

Suppose that we have residuals u_t from regression $Y_t = c + \phi Y_{t-1} + u_t$. Consider

$$u_t = \sqrt{h_t} v_t, \quad h_t = \zeta + \alpha u_{t-1}^2$$

where v_t is i.i.d $N(0,1)$. Find the log-likelihood function for (ζ, α).
Solution: Note that $u_t | u_{t-1} \sim N(0, \zeta + \alpha u_{t-1}^2)$. Then

$$L(\zeta, \alpha) = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi(\zeta + \alpha u_{t-1}^2)}} e^{-\frac{u_t^2}{2(\zeta + \alpha u_{t-1}^2)}}$$

The log-likelihood function

$$\Lambda(\zeta, \alpha) = -\frac{T-1}{2} \log(2\pi) - \frac{1}{2} \sum_{t=2}^{T} \log(\zeta + \alpha u_{t-1}^2) - \sum_{t=2}^{T} \frac{u_t^2}{2(\zeta + \alpha u_{t-1}^2)} \tag{16}$$

Test of Homoskedasticity vs ARCH(p)

- Consider

$$u_t^2 = \zeta + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \cdots + \alpha_p u_{t-p}^2 + w_t \qquad (17)$$

- Engle (1982): Lagrange Multiplier-type test
  - Regress Y_t on Y_{t-1} to get $\hat{u}_t$
  - Regress $\hat{u}_t^2$ on $\hat{u}_{t-1}^2, \cdots, \hat{u}_{t-p}^2$.
  - Get R^2 from this regression
  - $T \cdot R^2 \sim \chi^2(m)$

Testing Nested GARCH(p,q) Models

- Likelihood ratio test
- Estimating GARCH(1,1), get log-likelihood function $\Lambda(\theta_0)$
- Estimating GARCH(2,2), get log-likelihood function $\Lambda(\theta_1)$
- Note that GARCH(1,1) is a special case of GARCH(2,2) (take $\alpha_2 = \delta_2 = 0$)
- Hence $\Lambda(\theta_0) < \Lambda(\theta_1)$
- Likelihood ratio:

$$LR = 2(\Lambda(\theta_1) - \Lambda(\theta_0)) \sim \chi^2(2) \tag{18}$$

Estimating ARCH(1) Using Python

- Python package: `arch`
- Sample code:

```
model = arch_model(ret_data, mean='AR', lags=1, p=1,
    q=0)
am = model.fit()
```

- Equivalently:

```
model = arch_model(ret_data, mean='AR', lags=1,
    vol='ARCH', p=1)
am = model.fit()
```

- GARCH(1,1): Set $q = 1$