MGTF 411 Final Review Session

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May 29, 2025

Questions

- Review all the stuff included in the exam plz
  - Sure
- could you combine some example questions to show us how to use the formula and calculation?
  - Yes, I will.
- Can we have ten review sessions please :)
  - I am sorry I can't
- Other Questions?

Announcement

- Final Exam Time: Wednesday, June 4th, 2pm-5pm
- Location: WFN 4N128
- You are allowed to bring **10 double-sided** study sheets.
- Multiple choices and calculations. Roughly 25 questions.
- All lecture slides will be covered.

Binomial Models

- The world has two states at $t = 1$: $S_u = uS_0$ and $S_d = dS_0$
- Risk free rate is r_b. At time $t = 1$ you will earn r_b if you invest \$1 today
- The model is arbitrage free if (and only if) $d < r_b < u$
- Risk neutral measure in this model:

$$q = \frac{r_b - d}{u - d}$$

- Replication portfolio: Consider a call option with payoff $(S_1 - K)_+$ at $t = 1$
- Assume that $uS_0 > K > dS_0$

Replication Portfolio

- Call option payoff:

$$C_u = uS_0 - K, \quad C_d = 0$$

- There is no arbitrage $\Rightarrow$ LOOP
- Construct replication portfolio (θ_s, θ_b) such that the portfolio will have same payoff as the call option at $t = 1$:

$$\theta_s uS_0 + \theta_b r_b = uS_0 - K$$
$$\theta_s dS_0 + \theta_b r_b = 0$$

- Solution:

$$\theta_s = \frac{uS_0 - K}{(u-d)S_0}, \quad \theta_b = -\frac{1}{r_b} dS_0$$

FTAP

- I apologize for not completing the handout 1, which will cover the Fundamental Theorem in Asset Pricing I and II under discrete time framework.
- I will give illustrations here
- Risk neutral measure $\mathbb{Q}$: (1) Equivalent to physical measure $\mathbb{P}$ (2) Under $\mathbb{Q}$, all assets expected return are given by risk free rate. (3) $\mathbb{Q}(\omega_i) > 0$ (strictly greater than 0)

Theorem (FTAP I)

No arbitrage $\Leftrightarrow$ *Existence of risk neutral measure*

An Concrete Example

Consider the market with 1 risky asset and 1 risk less asset with $S_0 = 5$, $r_b = \frac{10}{9}$. But, there are three states in the world at $t = 1$, u, m and d.

$$S_u = \frac{60}{9}, \quad S_m = \frac{50}{9}, \quad S_d = \frac{30}{9}$$

Does this market have arbitrage opportunities?

Solution: We must have

$$\frac{1}{r_b}\mathbb{E}^{\mathbb{Q}}\left[\frac{S_1}{S_0}\right] = 1 \Rightarrow \frac{6}{5}q_u + q_m + \frac{3}{5}q_d = 1$$

Also, we must have $q_u + q_m + q_d = 1$. Solving this system will give you

$$\mathcal{Q} = \{\mathbb{Q} = (2c, 1-3c, c) | 0 < c < \frac{1}{3}\}$$

FTAP II

We say that markets are complete iff all $\mathcal{F}_T$-measurable random variables C_T can be replicated by a trading strategy.

Theorem (FTAP II)

Markets are complete iff the risk-neutral measure is unique.

From previous example we can see that $\mathbb{Q}$ is not unique. The the market is incomplete. Also, the 'risk-neutral price' for options are not unique.

Old Friend - BM

- A Brownian motion (B.M.) is a continuous stochastic process that satisfies the followings:
  - $B_0 = 0$
  - B_t has stationary and independent increments. E.g. $B_{t_2} - B_{t_1}$ is independent of $B_{t_3} - B_{t_2}$ if $t_1 \leq t_2 \leq t_3$.
  - $B_t - B_s \sim N(0, t-s), \forall t > s$
- $B_t - B_s \perp \mathcal{F}_s, t > s$
- B_t is a martingale.
- Moment generating function for normal distribution: $X \sim N(\mu, \sigma^2)$

$$\mathbb{E}[e^{\lambda X}] = e^{\mu\lambda + \frac{1}{2}\sigma^2\lambda^2}$$

MGF - Application

Suppose that B_t is a standard Brownian motion, calculate $\mathbb{E}[e^{\mu t+\sigma B_t}]$

Solution: Apply the MGF

$$\mathbb{E}[e^{\mu t+\sigma B_t}] = e^{\mu t}\mathbb{E}[e^{\sigma B_t}] = e^{\mu t}e^{\frac{1}{2}\sigma^2 t} = e^{(\mu+\frac{1}{2}\sigma^2)t}$$

Stochastic Integral

- For deterministic function of t, $\int_0^T f(t)\mathrm{d}B_t$ is a martingale.
- Itô Isometry:

$$\mathbb{E}\left[\left(\int_0^T f(t, B_t)\mathrm{d}B_t\right)^2\right] = \int_0^T \mathbb{E}[f^2(t, B_t)]\mathrm{d}t$$

- Example

$$\mathrm{Var}\left(\int_0^T t\mathrm{d}B_t\right) = \int_0^T t^2\mathrm{d}t = \frac{1}{3}T^3$$

Itô's Lemma

- You have to keep in mind... Some differentiation rule for stochastic calculus
- $\mathrm{d}(X_t \pm Y_t) = \mathrm{d}X_t \pm \mathrm{d}Y_t$
- $\mathrm{d}(X_t Y_t) = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t + (\mathrm{d}X_t)(\mathrm{d}Y_t)$
- $\mathrm{d}t\mathrm{d}t = 0, \mathrm{d}t\mathrm{d}B_t = 0, \mathrm{d}B_t\mathrm{d}B_t = \mathrm{d}t$
- Itô's Lemma:

$$\mathrm{d}f(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(\mathrm{d}X_t)^2$$

Exercise: Itô's Lemma

Suppose that $\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t$. Calculate $\mathrm{d}(X_t^n)$ for $n \geq 3$ by Itô's Lemma.

Solution:

$$\begin{aligned}
\mathrm{d}(X_t^n) &= nX_t^{n-1}\mathrm{d}X_t + \frac{1}{2}n(n-1)X_t^{n-2}(\mathrm{d}X_t)^2 \\
&= nX_t^{n-1}\mathrm{d}X_t + \frac{1}{2}n(n-1)\sigma^2 X_t^{n-2}\mathrm{d}t \\
&= X_t^{n-2}\left[\mu n X_t + \frac{1}{2}\sigma^2 n(n-1)\right]]\mathrm{d}t + n\sigma X_t^{n-1}\mathrm{d}B_t
\end{aligned}$$

BS Equation

- BS equation is nothing but a way of saying the portfolio is self-financing.

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

  (with some terminal conditions)
- It is **not** equivalent to no arbitrage (this requires deeper knowledge... not covered here)
- Wait... but TA, what is self-financing?

Self-Financing Portfolio

- Change in portfolio return are fully due to price change. No outside source of money.
- A portfolio $W_t = b_t\beta_t + n_tS_t$ is said to be self-financing if

$$\mathrm{d}W_t = b_t\mathrm{d}\beta_t + n_t\mathrm{d}S_t$$

- Consider the portfolio: $f(t, S_t) = tS_t$. Is this portfolio self-financing?
- Note that

$$\mathrm{d}(tS_t) = S_t\mathrm{d}t + t\mathrm{d}S_t \neq t\mathrm{d}S_t$$

- Not self-financing.
- You can also use BS equation to check whether the portfolio is self financing or not.

Risk Neutral Pricing

- Recall our previous discussion for FTAP I
- Suppose that there is no arbitrage opportunity, then the risk neutral measure exists.
- Every asset can be priced using

$$V_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}V_T]$$

- Under BS assumptions
- Consider a contingent claim pays you $X_T = S_T^3$ at time T. What should be the price of this contingent claim today (at time t)?
- Use the formula:

$$X_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}X_T] = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}S_T^3]$$

- Probability Theory.

Exercise: General Case

Suppose there is a contingent claim pays S_T^n at maturity T. The riskless rate is r. Find the price of this contingent claim.

Solution: By risk neutral pricing:

$$
\begin{aligned}
V_t &= S_t^n \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r(T-t)}e^{n(r-\frac{1}{2}\sigma^2)(T-t)+n\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}})}\right] \\
&= S_t^n e^{-r(T-t)}e^{n(r-\frac{1}{2}\sigma^2)(T-t)}\mathbb{E}_t^{\mathbb{Q}}[e^{n\sigma(B_T^{\mathbb{Q}}-B_t^{\mathbb{Q}})}] \\
&= S_t^n e^{-r(T-t)}e^{n(r-\frac{1}{2}\sigma^2)(T-t)}e^{\frac{1}{2}n^2\sigma^2} \\
&= S_t^n e^{(n-1)r(T-t)+\frac{1}{2}n(n-1)\sigma^2(T-t)}
\end{aligned}
$$

Stock for the long run

- Basic model:

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

- Log Return

$$\ln S_T - \ln S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) + \sigma B_T$$

has variance $\sigma^2 T$

- Note that

$$\frac{\ln S_T - \ln S_0}{T} = \mu - \frac{1}{2}\sigma^2 + \sigma \frac{1}{T}\int_0^T \mathrm{d}B_t = \mu - \frac{1}{2}\sigma^2 + \sigma \frac{B_T}{T}$$

has variance $\frac{\sigma^2}{T}$

Stock for the Long Run

- Recall what you got from the homework
- As a function of T, ϕ^* can be increasing or decreasing. It depends on the γ
- Think about the intuition behind the stock for the long run
- When time horizon T is increasing, both (*portfolio*) volatility and expected return are increasing.
- For less risk-averse individuals, they will invest more on stocks
- For more risk-averse individuals, they will invest less on stocks

Warren Buffet's Bet & CCAPM

- Take a look at slides for this part
- If you don't have time, at least know that the probability of he wins is increasing with horizon.
- For CCAPM, you need to know

$$r = \beta + \gamma \mu_c - \frac{1}{2}\gamma(\gamma + 1)\sigma_c^2$$

- When there is consumption risk, then the interest rate is lower.