MGTF 411 Handout 2: Brownian Motion

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1 Definition and Properties

Definition 1. *A Brownian motion (B.M.) is a continuous stochastic process that satisfies the followings*[1]*:*

- $B_0 = 0$
- B_t *has stationary and independent increments. E.g.* $B_{t_2} - B_{t_1}$ *is independent of* $B_{t_3} - B_{t_2}$ *if* $t_1 \leq t_2 \leq t_3$.
- $B_t - B_s \sim N(0, t-s), \forall t > s$

From the definition, we can easily see that $B_t \sim N(0, t)$.

Example 1. *Calculate* $\mathbb{E}[B_s B_t]$ *for* $t > s$

Solution: We have

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s(B_t - B_s + B_s)] = \mathbb{E}[B_s \underbrace{(B_t - B_s)}_{\text{independent of } B_s}] + \mathbb{E}[B_s^2] = s$$

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1.1 Hölder Continuity

Definition 2. *We say that a function* $f(t)$ *is Hölder continuous with index* $\alpha > 0$ *at point* t *if there exists a constant* C *such that*

$$|f(t) - f(s)| \leq C|t-s|^\alpha$$

Theorem 1. *Brownian motion is Hölder continuous with index* $\alpha < \frac{1}{2}$

Theorem 2. *Brownian motion is nowhere differentiable with probability 1.*

[1] Actually the construction of Brownian motion may be much more complex than you think. For example, does such a stochastic process exist? If you really want to know deeper knowledge behind this, please take MATH 280 series.

1.2 Reflection Principle

Theorem 3. *Suppose that $\{B_t\}_{t\geq 0}$ is a standard Brownian motion, then for any $a > 0$,*

$$\mathbb{P}(\max_{0\leq s\leq t} B_s \geq a) = 2\mathbb{P}(B_t \geq a)$$

Proof. Let $\tau_a := \min\{t : B_t = a\}$. Then

$$\mathbb{P}(B_t \geq a|\tau_a \leq t) = \frac{1}{2}$$

On the other hand, note that

$$\mathbb{P}(B_t \geq a, \tau_a \leq t) = \mathbb{P}(B_t \geq a)$$

Hence, by the first equation

$$\mathbb{P}(B_t \geq a) = \frac{1}{2}\mathbb{P}(\tau_a \leq t) = \frac{1}{2}\mathbb{P}(\max_{0\leq s\leq t} B_s \geq a)$$

□

2 Martingale

2.1 Filtration

Intuitively (and informally), a filtration $\{\mathcal{F}_t\}$ can be considered as increasing information set and no information is ever forgotten, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$, $\forall t > s$.

Definition 3. *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a continuous and increasing collection of σ-algebra $\{\mathcal{F}_t : t \geq 0\}$ is called a filtration if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $t > s$.*

Definition 4. *A filtration generated by stochastic process $\{X_t : t \geq 0\}$ is the collection:*

$$\mathcal{F}_t := \sigma\left(\bigcup_{0\leq s\leq t} \sigma(X_s)\right)$$

2.2 Martingale

Definition 5. $\{X_t\}_{t\geq 0}$ *is called a martingale with respect to $\mathcal{F}_t$ if*

- *X_t is integrable for each t*
- *X_t is adapted to $\mathcal{F}_t$. (X_t is $\mathcal{F}_t$ measurable for all t)*
- $\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \forall t \geq s$

In the context of Brownian motions, we generally define $\mathcal{F}_t = \sigma(\{B_s : s \leq t\})$. Naturally, B_t is $\mathcal{F}_t$-measurable. It is also easy to verify that B_t, $B_t^2 - t$ are martingales. I will present a more challenging example below:

Example 2. *Prove that* $X_t = \exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right)$ *is a martingale.*

Solution: Recall the moment generation function of normal distribution: suppose that $X \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Now for $t \geq s$,

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{F}_s] &= e^{-\frac{\lambda^2}{2}t}\mathbb{E}\left[e^{\lambda(B_t - B_s)}e^{\lambda B_s}|\mathcal{F}_s\right] \\ &= e^{-\frac{\lambda^2}{2}t}e^{\lambda B_s}\mathbb{E}\left[e^{\lambda(B_t - B_s)}|\mathcal{F}_s\right] \\ &= e^{-\frac{\lambda^2}{2}t}e^{\lambda B_s}\mathbb{E}\left[e^{\lambda(B_t - B_s)}\right] \\ &= e^{-\frac{\lambda^2}{2}t}e^{\lambda B_s}e^{\frac{\lambda^2}{2}(t-s)} \\ &= e^{\lambda B_s - \frac{\lambda^2}{2}s} = X_s\end{aligned}$$

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3 Quadratic Variation

Consider a partition of $[0, T]$:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$$

Also denote $\Delta t := t_{i+1} - t_i = \frac{T}{n}$. We define the quadratic variation of $\{X_t\}_{t \in [0,T]}$ as

$$[X_t, X_t] := m.s. \lim_{n\to\infty} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2 \tag{1}$$

Here the notation $m.s. \lim_{n\to\infty}$ means "mean-square" convergence, i.e. L^2-convergence.

Example 3. *Prove that* $[B_t, B_t] = T$

Solution: Note that

$$\mathbb{E}\left[\sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2\right] = \sum_{i=0}^{n-1} \Delta t = T$$

And

$$\text{Var}\left[\sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2\right] = \sum_{i=0}^{n-1} \underbrace{\text{Var}((B_{t_{i+1}} - B_{t_i})^2)}_{\text{Use } \chi^2} = \sum_{i=0}^{n-1} 2(\Delta t)^2 = \frac{2T}{n} \xrightarrow[n\to\infty]{} 0$$

Hence

$$\sum_{i=0}^{n-1}(B_{t_{i+1}}-B_{t_i})^2 \xrightarrow[n\to\infty]{L^2} T$$

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What about the first order variation? Note that if $X \sim N(\mu,\sigma^2)$, then $\mathbb{E}[|X|] = \sigma\sqrt{2/\pi}$. Then

$$\mathbb{E}\left[\sum_{i=0}^{n-1}|B_{t_{i+1}}-B_{t_i}|\right] = \sum_{i=0}^{n-1}\mathbb{E}|B_{t_{i+1}}-B_{t_i}|] = \sum_{i=0}^{n-1}\sqrt{\Delta t}\sqrt{\frac{2}{\pi}} = \sqrt{\frac{2}{\pi}}\sqrt{nT} \to \infty$$

Hence the first order variation of Brownian motion is unbounded.

What about a deterministic function of t? Consider $f(t) \in C^1$, then

$$[f(t),f(t)]_{[0,T]} = \lim_{n\to\infty}\sum_{i=0}^{n-1}(f(t_{i+1})-f(t_i))^2 = \lim_{n\to\infty}\sum_{i=0}^{n-1}[f'(\xi_i)\Delta t]^2 = \lim_{n\to\infty}\frac{T}{n}\int_0^T[f'(t)]^2\mathrm{d}t = 0$$

You might have seen the notations before: $(\mathrm{d}B_t)^2 = \mathrm{d}B_t\mathrm{d}B_t = \mathrm{d}t$, $\mathrm{d}t\mathrm{d}B_t = 0$, $\mathrm{d}t\mathrm{d}t = 0$. These notation basically mean the quadratic variation of B_t over $[0,T]$ is T, and it is equivalent to integrating $\mathrm{d}t$ over $[0,T]$.

4 Integrating BM over Time

Now we consider the following stochastic process:

$$Z_t = \int_0^t B_s\mathrm{d}s$$

The goal of this section is to derive the distribution of Z_t using definition of integral. Consider the Riemann sum:

$$\begin{aligned}
Z_t^{(n)} &:= \sum_{i=1}^{n} B_{t_i}\Delta t \\
&= (B_{t_1} + (B_{t_2} - B_{t_1} + B_{t_1}) + (B_{t_3} - B_{t_2} + B_{t_2} - B_{t_1} + B_{t_1}) + \cdots)\Delta t \\
&= (nB_{t_1} + (n-1)(B_{t_2} - B_{t_1}) + \cdots + 2(B_{t_{n-1}} - B_{t_{n-2}}) + (B_{t_n} - B_{t_{n-1}}))\Delta t \\
&\sim N\left(0, (\Delta t)^2\sum_{i=1}^{n} i^2(\Delta t)\right) = N\left(0, \frac{n(n+1)(2n+1)}{6}\frac{t^3}{n^3}\right) \xrightarrow{d} N\left(0, \frac{t^3}{3}\right)
\end{aligned}$$

Note that $Z_t^{(n)} \xrightarrow{d} Z_t$, hence $Z_t \sim N(0, t^3/3)$.