MGTF 411 Handout 3: Stochastic Calculus and SDE

Jiahui Shui

April 25, 2025

1 Stochastic Integration

1.1 Definition

Consider a partition of $[0, T]$: $0 = t_0 < t_1 < \cdots < t_N = T$. Recall the definition of Riemann integral for $f(t)$:

$$\int_0^T f(t)\mathrm{d}t := \lim_{N\to\infty} \sum_{i=0}^{N-1} f(\xi_i)\Delta t, \quad \xi_i \in [t_i, t_{i+1}]$$

In the definition, ξ_i can be any point within $[t_i, t_{i+1}]$. But for Itô integral with respect to Brownian motion, we consider the left endpoint of each interval:

$$\int_0^T f(t)\mathrm{d}B_t = \text{m.s.} \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, B_{t_i})(B_{t_{i+1}} - B_{t_i}) \tag{1}$$

Example 1. *Use definition, calculate*

$$\int_0^T B_t \mathrm{d}B_t$$

Solution: By definition, we have

$$\begin{aligned}
\int_0^T B_t \mathrm{d}B_t &= \lim_{N\to\infty} \sum_{i=0}^{N-1} B_{t_i}(B_{t_{i+1}} - B_{t_i}) \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} (B_{t_i} - B_{t_{i+1}} + B_{t_{i+1}})(B_{t_{i+1}} - B_{t_i}) \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} B_{t_{i+1}}(B_{t_{i+1}} - B_{t_i}) - \lim_{N\to\infty} \sum_{i=0}^{N-1} (B_{t_{i+1}} - B_{t_i})^2 \\
&= \lim_{N\to\infty} \sum_{i=0}^{N-1} B_{t_{i+1}}^2 - \lim_{N\to\infty} \sum_{i=0}^{N-1} B_{t_{i+1}} B_{t_i} - T
\end{aligned}$$

The last equality is due to the quadratic variation of B_t on $[0, T]$ is T. On the other hand, from the first equality, we have

$$\int_0^T B_t \mathrm{d}B_t = \lim_{N\to\infty} \sum_{i=0}^{N-1} B_{t_i} B_{t_{i+1}} - \lim_{N\to\infty} \sum_{i=0}^{N-1} B_{t_i}^2$$

Therefore,

$$2\int_0^T B_t \mathrm{d}B_t = B_T^2 - B_0^2 - T \Rightarrow \int_0^T B_t \mathrm{d}B_t = \frac{1}{2}B_T^2 - \frac{1}{2}T$$

■

We can also represent above equation in "differential" form:

$$\mathrm{d}(B_t^2) = 2B_t \mathrm{d}B_t + \mathrm{d}t$$

Whenever you see a differential form, you should interpret it as integrals:

$$\int_0^T \mathrm{d}(B_t^2) = 2\int_0^T B_t \mathrm{d}B_t + \int_0^T \mathrm{d}t$$

Similarly, one can define stochastic integral w.r.t X_t as

$$\int_0^T f(t, B_t)\mathrm{d}X_t = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, B_{t_i})(X_{t_{i+1}} - X_{t_i})$$

1.2 Deterministic Function

We first investigate the properties of following integral:

$$Z_T := \int_0^T f(t)\mathrm{d}B_t$$

where $f(t)$ is deterministic and $f \in L^2([0, T])$. We have

Theorem 1. *Z_t is normally distributed. $Z_T \sim N(0, \int_0^T f^2(t)\mathrm{d}t)$*

Proof. Note that

$$Z_T := \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)(B_{t_{i+1}} - B_{t_i})$$

$B_{t_{i+1}} - B_{t_i}$ are i.i.d normally distributed. Hence the sum is also normally distributed. To formally see this, one can try moment generating functions for $X_{t_i} := f(t_i)(B_{t_{i+1}} - B_{t_i})$, then

$$M_{X_{t_i}}(u) = \mathbb{E}[e^{uX_{t_i}}] = e^{\frac{1}{2}u^2 f^2(t_i)\Delta t}$$

Therefore,

$$M_Z(u) = \lim_{N\to\infty} \prod_{i=0}^{N-1} M_{X_{t_i}}(u) = \lim_{N\to\infty} \exp\left\{\frac{1}{2}u^2 \sum_{i=0}^{N-1} f^2(t_i)\Delta t\right\} = e^{\frac{1}{2}u^2 \int_0^T f^2(t)\mathrm{d}t}$$

Therefore,

$$Z_T \sim N\left(0, \int_0^T f^2(t)\mathrm{d}t\right)$$

□

1.3 Properties of Stochastic Integral

Proposition 1. *1. (Additivity) For $f(t, B_t)$ and $g(t, B_t)$,*

$$\int_0^T [f(t, B_t) + g(t, B_t)]\mathrm{d}B_t = \int_0^T f(t, B_t)\mathrm{d}B_t + \int_0^T g(t, B_t)\mathrm{d}B_t$$

2. (Homogeneity) For a constant c

$$\int_0^T cf(t, B_t)\mathrm{d}B_t = c\int_0^T f(t, B_t)\mathrm{d}B_t$$

3. (Partition)

$$\int_0^T f(t, B_t)\mathrm{d}B_t = \int_0^S f(t, B_t)\mathrm{d}B_t + \int_S^T f(t, B_t)\mathrm{d}B_t$$

4. (Zero Mean) We have

$$\mathbb{E}\left[\int_0^T f(t, B_t)\mathrm{d}B_t\right] = 0$$

The propositions above are standard and easy to verify. We will introduce more important properties as following:

Proposition 2. *(Itô Isometry)*

$$\mathbb{E}\left[\left(\int_0^T f(t, B_t)\mathrm{d}B_t\right)^2\right] = \int_0^T \mathbb{E}[f^2(t, B_t)]\mathrm{d}t$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T f(t, B_t)\mathrm{d}B_t\right)^2\right] &= \mathbb{E}\left[\lim_{N\to\infty}\left(\sum_{i=0}^{N-1} f(t_i, B_{t_i})(B_{t_{i+1}} - B_{t_i})\right)^2\right] \\
&= \lim_{N\to\infty} \mathbb{E}[f^2(t_i, B_{t_i})](t_{i+t} - t_i) + 2\sum_{i<j} \mathbb{E}[f(t_i, B_{t_i})f(t_j, B_{t_j})(B_{t_{i+1}} - B_{t_i})]\mathbb{E}[B_{t_{j+1}} - \\
&= \int_0^T \mathbb{E}[f^2(t, B_t)]\mathrm{d}t
\end{aligned}$$

□

Proposition 3. *(Covariance)*

$$\mathbb{E}\left[\left(\int_0^T f(t, B_t)\mathrm{d}B_t\right)\left(\int_0^T g(t, B_t)\mathrm{d}B_t\right)\right] = \int_0^T \mathbb{E}[f(t, B_t)g(t, B_t)]\mathrm{d}t$$

2 Differentiation & Itô's Lemma

We denote $\mathrm{d}X_t = X_{t+\mathrm{d}t} - X_t$. And this the differential form should be understood as stochastic integrals. Some basic rules of differentiation:

1. $\mathrm{d}(cX_t) = c\mathrm{d}X_t$, for any constant c
2. $\mathrm{d}(X_t + Y_t) = \mathrm{d}X_t + \mathrm{d}Y_t$
3. **Product Rule**: $\mathrm{d}(X_tY_t) = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t + (\mathrm{d}X_t)(\mathrm{d}Y_t)$
4. $\mathrm{d}t\mathrm{d}t = 0$,$\mathrm{d}t\mathrm{d}B_t = \mathrm{d}B_t\mathrm{d}t = 0$, $\mathrm{d}B_t\mathrm{d}B_t = \mathrm{d}t$

Theorem 2. *Suppose that*

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t$$

Then for a twice differentiable function $f(t, X_t)$, we have

$$\begin{aligned}
\mathrm{d}f(t, X_t) &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2 \\
&= \left(\frac{\partial f}{\partial t} + \mu(t, X_t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \sigma(t, X_t)\frac{\partial f}{\partial x}\mathrm{d}B_t
\end{aligned}$$

3 Some Techniques

3.1 Fundamental Theorem of Stochastic Calculus

Consider $\mathrm{d}X_t = f(t, B_t)\mathrm{d}B_t$, then we must have

$$X_t - X_s = \int_s^t f(u, B_u)\mathrm{d}B_u$$

An alternative definition to this is

$$\mathrm{d}X_t = \mathrm{d}\left(\int_s^t f(u, B_u)\mathrm{d}B_u\right)$$

Example 2. *Verify that*

$$\int_0^t B_s\mathrm{d}B_t = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

Solution: We only need to verify that

$$B_t\mathrm{d}B_t = \frac{1}{2}\mathrm{d}(B_t^2 - t)$$

which is easily obtained from Itô's lemma. ∎

Similarly, please verify that

$$\int_0^t sB_s\mathrm{d}B_t = \frac{t}{2}(B_t^2 - t) - \frac{1}{2}\int_0^t B_s^2\mathrm{d}s$$

3.2 Integration by parts

Suppose $f(t)$ is a deterministic function of t and $g(B_t)$ is a function of B_t. Then

$$\int_a^b f(t)g'(B_t)\mathrm{d}B_t = f(t)g(B_t)\Big|_a^b - \int_a^b \left[f'(t)g(B_t) + \frac{1}{2}f(t)g''(B_t)\right]\mathrm{d}t$$

Here is an example:

$$\int_0^t s\mathrm{d}B_s = sB_t\Big|_0^s - \int_0^t B_s\mathrm{d}s = tB_t - \int_0^t B_s\mathrm{d}s$$

In previous handout, we have already shown that $\int_0^t B_s\mathrm{d}s \sim N(0, \frac{t^3}{3})$.

4 Stochastic Differential Equations

We actually have seen SDE before:

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t$$

It can also be interpreted as

$$X_t - X_0 = \int_0^t \mu(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}B_s$$

Example 3. *Suppose that*

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}\mathrm{d}B_s$$

Then Itô's lemma tells us

$$\mathrm{d}X_t = \frac{b - X_t}{1-t}\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = a$$

Then a natural question is: Can we inverse the step above? i.e. Obtaining X_t from its SDE. In some cases, we do not need or we can not solve the SDE. In this section we will introduce some solvable cases.

4.1 Expectation

If we only want to know that expectation of the process, then most of time we do not need to fully solve the SDE. Instead, we only have to solve a ODE for the expectation, which is much simpler than solving SDE directly. That is

$$\mathbb{E}[X_t] = X_0 + \int_0^t \mathbb{E}[\mu(s, X_s)]\mathrm{d}s$$

Also, we can do the inverse to find the expectation. Consider the following example: Calculate $\mathbb{E}[B_t e^{B_t}]$. Note that

$$\mathrm{d}(B_t e^{B_t}) = \frac{1}{2}(B_t + 2)e^{B_t}\mathrm{d}t + \frac{1}{2}(B_t + 1)e^{B_t}\mathrm{d}B_t$$

Then

$$\begin{aligned}\mathbb{E}[B_t e^{B_t}] &= \frac{1}{2}\int_0^t \mathbb{E}[(B_s + 2)e^{B_s}]\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \mathbb{E}[B_s e^{B_s}]\mathrm{d}s + \int_0^t \mathbb{E}[e^{B_s}]\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \mathbb{E}[B_s e^{B_s}]\mathrm{d}s + 2(e^{\frac{t}{2}} - 1)\end{aligned}$$

Let $\varphi(t) = \mathbb{E}[B_t e^{B_t}]$, then

$$\varphi'(t) = \frac{1}{2}\varphi(t) + e^{\frac{t}{2}}, \quad \varphi(0) = 0 \Rightarrow \varphi(t) = te^{\frac{t}{2}}$$

4.2 Exact Stochastic Equations

Recall that for $f(t, B_t)$,

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \frac{\partial f}{\partial x}\mathrm{d}B_t$$

Suppose that there exists a function $f(t, B_t)$ such that

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2} = \mu(t, x) \\ \dfrac{\partial f}{\partial x} = \sigma(t, x) \end{cases}$$

Then

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t$$

is called an exact equation. The solution is then given by $X_t = f(t, B_t) + c$. The condition for the equation to be exact is

$$\frac{\partial \sigma}{\partial t} + \frac{1}{2}\frac{\partial^2 \sigma}{\partial x^2} = \frac{\partial \mu}{\partial x}$$

4.3 Linear Stochastic Differential Equation

Consider the following SDE:

$$\mathrm{d}X_t = (\alpha(t)X_t + \beta(t))\mathrm{d}t + \sigma(t, B_t)\mathrm{d}B_t$$

One can mimic the integrating factor approach for ODE to obtain the solution:

$$X_t = X_0 e^{A_t} + \int_0^t e^{A(t)-A(s)}\beta(s)\mathrm{d}s + \int_0^t e^{A(t)-A(s)}\sigma(s, B_s)\mathrm{d}B_s$$