Risk Seeking

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Abstract

People are risk-seeking in certain situations, though they are normally risk-averse. The loss aversion utility function provides such an example. Risk seeking is largely understudied, probably because it usually does not allow optimal choices and are not tractable. In this paper, we study the implications when risk seeking is incorporated into the agent's preferences. We show that risk seeking dramatically alters the agent's behaviors in stressed scenarios. It is optimal to take large long or short positions and shun positions involving moderate levels of risk. The agent can swing between sizable long and short positions with minor changes in market conditions. The agent may short an asset with a positive risk premium. These behaviors are consistent with findings in experimental and market settings but cannot be explained by risk-averse preferences.

Key words: Risk seeking, optimal choice, loss aversion, HARA.

JEL Classification: C61, G11

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1 Introduction

There are moments when humans exhibit risk-seeking behaviors, even though they are normally risk-averse. Risk-seeking behaviors have long been noted by Friedman and Savage (1948), Markowitz (1952), and Williams (1966), and featured in the loss aversion utility function developed in Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Risk seeking following losses is even observed in monkeys.[1] However, this permanent psychological attribute has been largely overlooked in the literature, probably because it may not allow optimal choices under uncertainty and is not quite tractable.

The objective of this paper is to study the implications of risk seeking for choices. To achieve this, we apply the loss aversion utility function to the classical portfolio choice problem.[2] This utility function, developed in monetary settings, is both parsimonious and capable of capturing the important psychological attribute: people are generally risk-averse but become risk-seeking in certain situations ("losses" in the context of loss aversion preferences). It represents a minimal deviation from traditional risk-averse utility functions and often yields optimal choices. Both investment and asset pricing problems involve such a choice. Our analytical results isolate the effects of risk seeking in loss aversion preferences. We show that risk seeking leads to outcomes that are consistent with findings in experimental and market settings but cannot be explained by risk-averse preferences.

We find that the agent's choice varies sharply across two scenarios, "underwater" and "above-water", in which the agent's initial wealth is low and high, respectively. Our paper focuses particularly on the underwater scenario, since in this scenario, risk seeking has significant effects on choices.

First, we show that in the underwater scenario, the sign of the optimal portfolio weight depends not on the sign of the risk premium, but on the "adjusted risk premium," which

[1] https://www.bbc.com/worklife/article/20180406-what-monkeys-can-teach-us-about-money.

[2] Loss aversion preferences have been extensively utilized to elucidate the behaviors of individuals and asset prices. Benartzi and Thaler (1995) find that loss aversion helps explain the equity premium puzzle due to the reluctance of agents to invest in stocks. Barberis, Huang and Santos (2001) show that loss aversion produces excess return volatility and low correlation between stock returns and consumption growth. Gomes (2005) and Barberis and Xiong (2009) use loss aversion to explain the disposition effect and low equity market participation rates. Li and Yang (2013) find that diminishing sensitivity in loss aversion predicts the disposition effect, price momentum, a reduced return volatility, and a positive return-volume correlation.

is the difference between the risk-adjusted expected return and the riskless rate. The agent shorts the risky asset with a negative adjusted risk premium, even when its risk premium is positive.[3] In fact, the agent under the water is eager for large returns to get out, placing less emphasis on risk. This can generate "risk-return doubledown," instead of a tradeoff. The above result further implies that the agent tends to short an asset with high return volatility because its risk-adjusted return is low.

Second, when the agent is under the water, her choices become "schizophrenic." Small changes in market conditions can cause the optimal portfolio weight to jump between local maxima. The schizophrenia arises from the varying dominance of risk-seeking and risk-averse behaviors across different portfolio weights. In particular, when the risk-adjusted expected return equals the riskless rate, there are two optimal portfolio weights—one positive and one negative—resulting in the agent being indifferent between strong leverage and shorting. These behaviors, shorting high-yield stocks and schizophrenia, stem from risk seeking. However, with a bilinear loss aversion utility function, which is well-examined in the literature, the agent shows local risk neutrality and global risk aversion, without any risk-seeking behavior. Consequently, both behaviors disappear.

Third, the agent takes large risky positions, either "stressed long" or "stressed short" (betting on positive or negative market conditions, respectively), in an attempt to move back into the gain domain. Intermediate positions are never optimal, since modest levels of risk would likely result in wealth in the loss domain. This further implies that the agent consistently participates in the stock market, even if the stock has a zero risk premium. In contrast, concave utility functions would result in zero holdings in this case. The non-zero holding is attributable to risk-seeking or risk-neutral behavior, but not risk-averse's. When the agent anticipates a high probability of future losses, she would likely choose to gamble, as it offers a chance to return to the gain domain, rather than taking no action.

When the agent's initial wealth is high (the "above-water scenario"), the agent behaves similarly to an agent with hyperbolic absolute risk aversion (HARA) preferences, though in a more aggressive manner. The sign of the optimal portfolio weight is the same as that of the

[3] By contrast, a risk-averse agent is always long this asset in a static setting. In a dynamic portfolio choice problem, a risk-averse agent (with a concave utility function) can short an asset with a positive risk premium for intertemporal hedging. For example, Liu (2001) shows that when the asset price follows the Heston model, where the risk premium is always positive, a CRRA agent can short this asset.

risk premium, rather than determined by the adjusted risk premium. The risky positions in this scenario are much smaller than those taken in the underwater scenario. These smaller risky positions serve to protect her gains and prevent her from falling into the loss domain while above water; however, they are insufficient to help her return to the gain domain when she is underwater. The significant shifts in choices between the underwater and above-water scenarios stem from the dominance of risk-seeking versus risk-averse behaviors. In the underwater scenario, the expected utility is primarily influenced by the loss component, whereas in the above-water scenario, it is driven by the gain component.

In our model, markets are incomplete, which limits the influence of risk-seeking behavior.[4] Li et al. (2024) demonstrate that in complete markets, which offer more investment opportunities, an agent under the water consolidates all losses into a single state. In such a scenario, the agent may choose to long an asset with a negative risk premium, in addition to shorting an asset with a positive risk premium, actions contrary to risk-averse predictions. In the above-water scenario, the optimal portfolio weight is always identical to that under HARA preferences. Comparing our findings with those in Li et al. (2024), the largest differences in optimal choices between complete and incomplete markets occur when the agent is under the water, while similar choices emerge when the agent is above the water.

A number of properties of the risk-seeking behavior can be consistent with empirical findings. Coval and Shumway (2005) find that following morning losses, professional market makers are far more likely to take on additional afternoon risk and trade (either buy or sell) more aggressively, which seem to be consistent with the schizophrenic behavior (bipolar choices) found in our paper. The large risky positions in the underwater scenario are consistent with individuals' risk-taking behavior in gambling even when the odds are not in their favor. For example, financial desperation appears to be an important driver of lottery participation (e.g., Beshears, Choi, Laibson and Madrian, 2018), and lower-income individuals demonstrate a higher propensity for lottery participation (e.g., Haisley, Mostafa and Loewenstein, 2008). The lottery participation by less wealthy agent, which is typically considered as irrational, could be justified by loss aversion utility.[5]

[4]Complete markets are explored in Berkelaar, Kouwenberg and Post (2004), Barberis and Xiong (2009), and Li, Liu and Shui (2024), among others, typically yielding more tractable results. Of these, Li et al. (2024) studies a static choice problem without portfolio constraints and thus is most closely related to our paper.

[5]The concept of probability weighting in prospect theory provides another rationale for lottery partici-

One major message delivered from our paper is that taking risk can be optimal under the expected utility framework.[6] Our results can be useful for firms in stressful situations, e.g., with corporate debt overhang, to devise optimal strategies. It has long been documented that firm managers exhibit risk-seeking behavior in response to below-target returns (e.g., Laughhunn, Payne and Crum, 1980), and troubled firms have a tendency to undertake greater risks (e.g., Bowman, 1982). More broadly, it could help governments to optimally overcome crises. For example, during the Global Financial Crisis in 2008, governments took on significant debt, which was risky, to prevent a severe and prolonged economic downturn.

Risk-seeking behavior, represented by the convex portion of the utility function, complicates the maximization process. First, it causes choices to be intrinsically global, and one cannot infer the utility's global properties from its local properties (e.g., FOCs). Numerical simulation methods as often used in the literature may fail to ensure optimality, potentially converging to local maxima or corner solutions. Second, bounded optimal policy may not always exist mathematically. Furthermore, we show that a small change in parameters can lead to portfolio jumps in three forms: a switch across the watermark, a shift between large long and short positions under the water, and a solution explosion due to the breakdown of global loss aversion. These inherent breaks with the optimal policy pose a great challenge to model and identify the agent's choices. In the existing literature, one approach to dealing with these challenges is to impose certain portfolio/wealth constraints or utility variations. To pinpoint the effects of risk seeking, our paper focuses on an unconstrained problem and examines the global behaviors inherently associated with risk seeking.[7] The global properties obtained in our paper help address the above challenges. Furthermore, our analytical results provide parameter restrictions for both the underlying assets and the utility function.

Risk seeking has been largely overlooked, even within the loss aversion literature. Loss aversion preferences encompass three key characteristics: evaluation of "losses" and "gains"

pation. It posits that individuals tend to overweight small probabilities, such as those of winning a lottery, driving people to purchase lottery tickets. This explanation is different from the mechanism of risk seeking, which predominantly comes into play during financial distress or when individuals are facing losses.

[6]Loss aversion preference by itself (without probability weighting) is consistent with expected utility theory (Ingersoll, 2024, Chapter I-13).

[7]The optimal policies under constraints are often given by corner solutions and largely reflect the constraints. Constraints also cause choices to be always bounded regardless of the degree of risk seeking, mitigating the effects of risk-seeking behavior.

relative to a reference point, known as *reference dependence*; greater sensitivity to losses than to equivalent gains, termed *loss aversion*; and *diminishing sensitivity*, showing risk aversion with gains but risk seeking with losses. Barberis (2013) observes that while reference dependence and loss aversion are useful in many applications of prospect theory, "*[d]iminishing sensitivity, by contrast, seems much less important.*" One potential reason for this observation is that the literature predominantly explores the above-water scenario,[8] which hides the effects of risk seeking. Another reason lies in the portfolio or wealth constraints often assumed in this literature. While these constraints have minimal impact on local characteristics, such as reference dependence and loss aversion, they substantially dampen the effects of risk seeking, a global property of the preferences. Furthermore, risk seeking is a unique feature of loss aversion preferences, absent in other widely used economic models. By contrast, the other two features—reference dependence and loss aversion—are shared across many preferences.[9]

The choice of the reference point is a key challenge in the application of prospect theory (Barberis, 2013).[10] We take the reference point as given but provide a general analysis of its effects on static decision-making for any specified reference level. We demonstrate that the most significant effect of the reference point is influencing the watermark, below or above which optimal policies vary greatly due to the differing dominance of risk-seeking and risk-averse behaviors. An adjustment in the reference point can lead the agent to transition from conservative investments to highly risky positions, crossing the watermark from above to below. Within each scenario, the effect of the reference point is fully captured by a reference adjustment factor, which proportionally alters the optimal portfolio weight.

The realization utility literature, e.g., Barberis and Xiong (2009), Ingersoll and Jin (2013),

[8]For example, one reference point frequently chosen in the literature is current wealth, which is consistent with the suggestions of Kahneman and Tversky (1979). With this choice, the agent is above-water.

[9]E.g., disappointment aversion (Gul, 1991), costly adjustment for living standards (Dybvig, 1995; Choi, Jeon and Koo, 2022), habit formation (Campbell and Cochrane, 1999), and consumption commitments (Chetty and Szeidl, 2016).

[10]The literature offers various choices for the reference point. Kahneman and Tversky (1979) suggest it is current wealth or expectations. Tversky and Kahneman (1991) argue it is influenced by aspirations, expectations, norms, and social comparisons. In financial markets, it can be the purchase price (Shefrin and Statman, 1985), the historical price peak (Gneezy, 2005), or the current price (Baucells, Weber and Welfens, 2011). Baillon, Bleichrodt and Spinu (2020) find the most common reference points are the status quo and a secure level representing the maximum achievable outcome.

and Dai, Qin and Wang (2024), posits that a utility burst is received only upon the realization of a gain or loss, a concept supported by mental accounting. This utility helps explain the disposition effect. The results are most significant in the context of the stock experiencing losses, where risk seeking plays a crucial role, akin to the underwater scenario. However, this literature typically examines constrained choice problems, such as wealth or leverage constraints and binary choices, which mitigate the effects of risk seeking. The three key predictions of our model are not addressed in this literature. For example, the agent in Dai et al. (2024) does not take sizable positions after loss realizations, probably because she faces a leverage constraint and is not required to spend entire budget when trades. Dai et al. (2024) find that the agent in their model, due to two-layered mental accounts, typically holds intermediate positions; in contrast, our paper shows that under pure risk-seeking behavior (without constraints and mental accounting), these positions are never optimal in the underwater scenario. In addition, both schizophrenia and shorting a positive-risk-premium asset are absent in their model. The risky positions bounded from below limit the agent to betting on bad states, which actually provide an opportunity to recover.

Dynamic reference points are also explored in Barberis et al. (2001), Köszegi and Rabin (2006), Meng and Weng (2018), among others, in addition to the studies on realization utility, and are found to significantly affect choices. There are numerous on-going discussions on the formation of reference points under and beyond loss aversion preferences (e.g., references in footnote 6), and their effects are still unsettled. Although we study a static setting, our results can be used to understand dynamic choices, since the dynamic problem within each rebalancing period is a static one.

He and Zhou (2011) study portfolio choice under prospect theory without constraints, primarily addressing solution boundedness, which necessitates parameter restrictions and is crucial for identifying applicable contexts for loss aversion. In contrast, our paper emphasizes the implications of risk-seeking behavior. Constrained problems with an exogenous reference point are explored in, e.g., Berkelaar et al. (2004) and Bernard and Ghossoub (2010), which focus on above-water or at-the-water scenarios. Conversely, our paper imposes no constraints and highlights the underwater scenario.

The executive compensation problem with a call option incentive (e.g., Carpenter, 2000; Ross, 2004) also involves non-concave utility functions. First, the expected utility in our paper is a portfolio of a long call and a short put, which is more involved than the call option incentive. Second, the objective function in Carpenter (2000) can be concavified without affecting the optimal policy; thus the choices are similar to, but more aggressive than, those under risk-averse preferences. However, for the loss aversion utility function studied in our paper, the convexity in the loss domain cannot be concavified, disconnecting its local and global properties and generating distinct differences from risk-averse preferences.

The paper is organized as follows. Section 2 discusses loss aversion preferences. Section 3 outlines the optimization problem and presents the optimal choices. Section 4 studies the properties of the optima choices, and Section 5 examines comparative statics. Section 6 concludes. Calculation details are included in the appendices.

2 Risk-Seeking Preferences

Risk-seeking behaviors have been extensively noted in the literature over a long history. To explain a significant class of individual reactions to risk, Friedman and Savage (1948) introduce a utility function of income, proposing that individuals exhibit risk-seeking behavior at mid-range income levels, while displaying risk aversion at both high and low income levels. Building on this, Markowitz (1952) introduces a four-segment utility function that is convex (risk-seeking) at low wealth levels and around current wealth levels, but concave in other regions.[11] Prospect theory, grounded in extensive experimental evidence, was originally introduced by Kahneman and Tversky (1979) and later expanded by Tversky and Kahneman (1992). This framework features a loss aversion utility function, suggesting that individuals tend to exhibit risk-seeking behavior in the domain of losses.

The above studies show that individuals exhibit risk-seeking behaviors in certain situations, even though they are normally risk-averse. Propensity for risk seeking after incurring losses is well-documented in both controlled laboratory studies (e.g., Andrade and Iyer, 2009) and natural experiments (e.g., Page, Savage and Torgler, 2014). Risk-seeking behavior in stressful situations is also observed in everyday life. Individuals facing serious illnesses are more inclined to take significant risks with aggressive treatments, whereas those with less severe conditions typically avoid such high-risk interventions. In football games, it is fre-

[11] This utility function has recently gained support from the experimental study with mixed prospects conducted by Levy and Levy (2002).

quently observed that the losing team adopts high-risk strategies in the final moments, such as having all players participate in the attack with minimal defense—a tactic not typically employed during regular play. Risk-seeking behavior following losses has been observed even in monkeys (see footnote 1).

However, this enduring psychological trait has been largely overlooked in the literature. One possible explanation is that in many scenarios, risk-seeking behavior does not readily lead to optimal choices, limiting its applicable contexts. Furthermore, analyzing such behavior requires a global perspective, making it less tractable.

2.1 Loss Aversion Preferences

To study the implications of risk-seeking preferences, we employ the loss aversion utility function developed by Kahneman and Tversky (1979) and Tversky and Kahneman (1992). This utility function is consistent with expected utility theory (Ingersoll, 2024) and represents a minimal deviation from risk-averse utility functions. It reduces to the hyperbolic absolute risk aversion (HARA) utility family in a limiting case, as shown shortly in Section 2.2. Additionally, it often yields optimal choices.

The los aversion utility function is defined over gains and losses relative to a reference point θ:

$$u(W) = \begin{cases} \frac{1}{1-\gamma}(W-\theta)^{1-\gamma} & \text{for} \quad W \geq \theta; \\ -A\frac{1}{1-\gamma}(\theta - W)^{1-\gamma} & \text{for} \quad W < \theta, \end{cases} \tag{1}$$

where W is the agent's wealth, $\gamma \in [0, 1)$ controls the curvature, and A measures the degree of loss aversion. Figure 1 illustrates the loss aversion utility function and shows that it is increasing with an S shape.

There are three key features of the loss aversion utility function (1). First, the utility function is concave in the gain domain $W > \theta$ and convex in the loss domain $W < \theta$ (the S-shaped utility function), a featured known as diminishing sensitivity. It implies that a loss-averse agent is risk-averse with gains but risk-seeking with losses.

The second feature, reference dependence, involves the agent evaluating deviations from a reference point θ, rather than focusing solely on the level of wealth. The third feature, loss aversion, refers to the phenomenon that people are more sensitive to losses than to equivalent gains. In this paper, we use "loss aversion" to denote this specific phenomenon, while "loss aversion preferences" will refer to the broader concept of risk assessment preferences. It leads to a kink of the utility function (1) at the reference point θ. We sometimes refer to this as "local loss aversion" to differentiate it from the global properties of loss aversion.[12] Due to the kink, first-order risk aversion (Segal and Spivak, 1990) (for $A > 1$) or first-order risk seeking (for $A < 1$) applies at $W = \theta$. For the other points, second-order risk aversion ($W > \theta$) or risk seeking ($W < \theta$) applies. This differs from Knightian uncertainty and disappointment aversion (Gul, 1991), with which the risk aversion is first-order at every level.[13]

Notably, the first feature, diminishing sensitivity, particularly the risk-seeking behavior it entails, is distinctly associated with the loss aversion preference and is not accounted for by other popular models of preferences. In fact, most utility functions used in economics are concave. However, the last two features (i.e., reference dependence and loss aversion) are also associated with other models, e.g., HARA utility, disappointment aversion (Gul, 1991), racheting of consumption (Dybvig, 1995), habit formation (Campbell and Cochrane, 1999), and consumption commitments (Chetty and Szeidl, 2016), among others.

Moreover, risk seeking is also overlooked within the loss aversion literature. This oversight arises probably from two key tendencies: First, this literature primarily addresses the "above-water" or "at-the-water" scenarios, effectively concealing the influence of risk-seeking

[12]Kahneman (2003) explained that "*The core idea of prospect theory [is] that the value function is kinked at the reference point and loss averse.*" This local property is used to define the loss aversion index in Köbberling and Wakker (2005) and called "loss aversion for small stakes" in Köszegi and Rabin (2006). On a larger scale, loss aversion can be a common feature of all concave utility functions following an affine transformation.

[13]Cumulative prospect theory developed in Tversky and Kahneman (1992) generally allows different curvature coefficients $\gamma_\pm \in [0, 1)$ over the gain and loss domains: $u(W) = \begin{cases} (W-\theta)^{1-\gamma_+} & \text{for} \quad W \geq \theta; \\ -A(\theta - W)^{1-\gamma_-} & \text{for} \quad W < \theta. \end{cases}$ When $\gamma_+ \neq \gamma_-$, this utility function exhibits diminishing sensitivity and reference dependence, and the degrees of risk seeking and risk aversion are separately governed by γ_- and γ_+. However, whether the investor is loss averse depends on the size of positions (Köbberling and Wakker, 2005; Bernard and Ghossoub, 2010). It cannot simultaneously exhibit local loss aversion around the reference point and assure global solutions (He and Zhou, 2011; Li et al., 2024). The utility function (1) with identical curvature coefficients over the gain and loss domains is estimated in Tversky and Kahneman (1992) and widely considered in the literature (e.g., Benartzi and Thaler, 1995). In this case, loss aversion behavior is completely controlled by coefficient A (when $A > 1$, the investor is always loss averse.) The boundedness of solution (we interpret it as "global loss aversion") in this case is also determined by A, as shown shortly in Lemma 1. As a result, the utility function (1) can simultaneously allow both local loss aversion and global loss aversion.

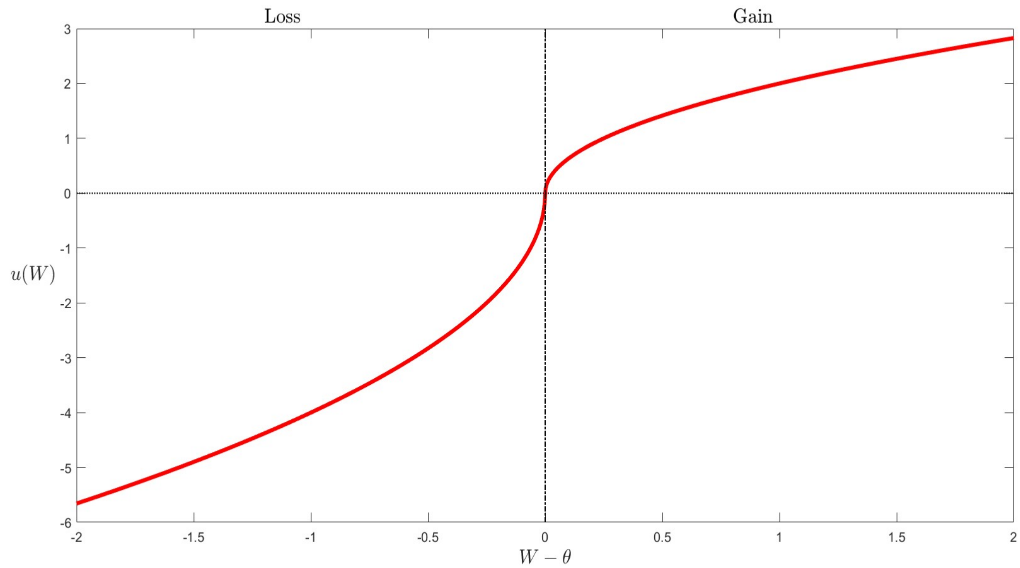

Figure 1: This figure illustrates the loss aversion utility function. Here, $A = 2$ and $\gamma = 0.5$.

behavior, which becomes prominent in the "underwater" scenario. Second, the portfolio or wealth constraints commonly assumed in this literature significantly mitigate the impact of risk-seeking preferences—a global characteristic—while having comparatively less effects on local features, such as reference dependence and loss aversion. In this paper, to examine the implications of risk seeking, we take the loss aversion utility of Tversky and Kahneman (1992) at face value without constraints.[14]

2.2 Relation between Loss Aversion and HARA Preferences

The loss aversion utility function is related to the HARA utility family, which was studied in Merton (1971). In the loss aversion utility function (1), coefficient A controls the penalty for losses. A larger A causes the agent to be more averse to losses. With $A \to +\infty$, (1) becomes

[14]We recognize that in the real world, individuals encounter diverse portfolio and wealth constraints, making the study of constrained portfolio problems equally crucial. In general, optimal policies with and without constraints are different. Li et al. (2024) show that constraints can qualitatively change the optimal policy under loss aversion. Imposing constraints is equivalent to redefining the utility function: it sets the utility to be minus infinity for wealth level beyond the constraints.

the HARA utility function:[15]

$$u(W) = \begin{cases} \frac{1}{1-\gamma}(W-\theta)^{1-\gamma}, & \text{for} \quad W \geq \theta; \\ -\infty, & \text{for} \quad W < \theta. \end{cases} \tag{2}$$

The HARA utility (2) is identical to the loss aversion utility with wealth above θ but different from the loss aversion utility with minus infinity utility when $W < \theta$.

The two types of preferences differ in several ways. First, the key distinction lies in their treatment of risk-seeking behavior, which is absent in HARA preferences. As demonstrated shortly, the most significant differences in optimal choices between loss aversion and HARA preferences stem from the presence of risk-seeking behavior under loss aversion. Second, Ingersoll (2016) defines that a utility function $u(\cdot)$ displays "weak loss aversion" if $u(W) + u(-W) \leq 0$, $\forall W > 0$. In this sense, the HARA utility also features loss aversion: when we measure "losses" and "gains" relative to θ, a HARA agent is more sensitive to losses than to equivalent gains (infinitely averse to losses). As a result, a HARA agent tends to be both more risk-averse and more loss-averse than a loss-averse agent.

Third, loss aversion preferences impose no restrictions on wealth levels, whereas HARA preferences lead to infinite marginal utility at the threshold θ. Consequently, in underwater scenarios, HARA utility functions are not well-defined, rendering them unsuitable for such cases. Additionally, in above-water scenarios, the optimal policy under HARA often results in corner solutions, as demonstrated in Section 3.4. Collectively, these results highlight the greater flexibility of loss aversion preferences compared to HARA in decision-making contexts. They further suggest that non-negative wealth constraints as used in the literature tend to impose more significant restrictions on the implications of loss aversion preferences.

The HARA utility function is uniformly concave, which allows one to infer global properties from local properties. FOCs are sufficient conditions for optimality. The optimal choice under HARA preferences is much simpler compared to that under loss aversion preferences. In fact, the convexity of the loss aversion utility function creates a disconnect between its local and global properties. This results in multiple local maxima and discontinuities in the optimal policy. Consequently, loss aversion preferences demand a global examination, as

[15] The HARA family is given by $u(W) = \frac{\gamma}{1-\gamma}(\frac{\beta W}{\gamma} + \eta)^{1-\gamma}$. Here we set $\theta \equiv -\frac{\gamma\eta}{\beta}$ and $\beta^{1-\gamma}\gamma^{\gamma} = 1$. With $\gamma \in [0, 1)$, the HARA utility function (2) has decreasing absolute risk aversion (DARA).

extrapolating global properties from local analyses becomes inherently challenging.[16]

3 Optimal Choices

In this section, we study the optimal choices under loss aversion preferences. We first describe the portfolio choice problem and the conditions under which the problem has bounded solutions. Under these conditions, we derive the optimal portfolio weights.

3.1 The Choice Problem

To study the properties of loss aversion as preferences of choice, we consider the classical portfolio choice problem, which could be a natural approach for this objective. Both investment and asset pricing problems involve such a choice.

There are two assets: a risky asset of which the gross return over a horizon of T is given by $R_T = e^{(\mu-\sigma^2/2)T+\sigma\sqrt{T}\epsilon}$, where ϵ is a standard normal random variable, and μ and σ are constant instantaneous expected return and volatility, and a riskless asset with a gross return over the same horizon given by $R_f = e^{r_f T}$, where r_f is a constant riskless rate. In this paper, we also refer to R_f as the riskless return for short. We consider a static portfolio choice problem over an investment horizon $[0, T]$, in which the agent maximizes

$$\max_{\phi} \mathbb{E}\big[u(W_T)\big], \tag{3}$$

where ϕ is the portfolio weight of the risky asset at time 0, and W_T is the end of period wealth satisfying

$$W_T = W_0\big[R_f + \phi(R_T - R_f)\big]. \tag{4}$$

The loss aversion preferences generally permit any level of wealth. Unless stated otherwise, we assume $W_0 > 0$. The results for negative initial wealth are symmetric to those for positive initial wealth, as demonstrated in Appendix C.

The $u(\cdot)$ in (3) is the agent's utility function. Instead of concave utility functions as widely examined in the literature, this paper assumes a loss aversion utility function (2).

[16]Numerical simulations over finite domains may fail to ensure optimality, as they could converge to local maxima, corner solutions, or even fail to produce finite solutions. Additionally, the lack of twice-differentiability of expected utility with respect to portfolio weights makes it harder to assess its convexity or concavity.

Other than the utility function, problem (3) is standard. In general, we do not obtain an explicit expression of the optimal portfolio weight in the static optimal problem (3), even under CRRA utility functions.

3.2 Boundedness Condition (Global Loss Aversion)

The loss aversion utility function (1) consists of two parts. The concave part over the gain domain tends to produce internal solutions, like standard risk-averse utility functions; however, the convex part over the loss domain typically leads to corner solutions and large positions. As a result, internal solutions may not always exist under (1). Lemma 1 states the criterions for bounded optimal portfolio weights.

Lemma 1. *(Boundedness criterion.) Define*

$$\underline{A} = \max\left\{\frac{\mathcal{C}}{\mathcal{P}}, \frac{\mathcal{P}}{\mathcal{C}}\right\}, \tag{5}$$

where

$$\mathcal{C} = \mathbb{E}\Big[\Big(\frac{R_T}{R_f} - 1\Big)^{1-\gamma}\mathbf{1}_{\{\frac{R_T}{R_f}\geq 1\}}\Big], \quad \mathcal{P} = \mathbb{E}\Big[\Big(1 - \frac{R_T}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{\frac{R_T}{R_f}< 1\}}\Big], \tag{6}$$

and $\mathbf{1}_S$ is the indicator function of set S.

1. *When $A > \underline{A}$, the optimal portfolio weight is bounded.*
2. *When $A < \underline{A}$, the optimal portfolio weight is unbounded.*
3. *When $A = \underline{A}$, the optimal portfolio weight is bounded for $\theta < W_0R_f$ and unbounded for $\theta > W_0R_f$, and any portfolio weights are indifferent for $\theta = W_0R_f$.*

Lemma 1 shows that the boundedness of solutions depends on the penalty level for losses imposed by the loss aversion utility. When the loss aversion coefficient A is small, the penalty for losses is small, and hence the agent tends to take an infinite (long or/and short) risky position. On the other hand, large A imposes large penalty for losses, which presents the expected utility from approaching positive infinity. In this paper, we interpret the boundedness condition in Lemma 1 as "global loss aversion": under this condition, the gain component in the expected utility is smaller than the loss component when portfolio weights are large in absolute value. The global loss aversion also significantly impacts global properties, such as monotonicity and curvature, of the expected utility, as shown in Corollary 6. Lemma 1 is in line with the general well/ill-posedness conditions developed in He and Zhou (2011).

In our model, markets are incomplete. Li et al. (2024) show that in complete markets, which allow more investment opportunities, the solution boundedness conditions become stricter than the incomplete-market case, and the lower bound $\underline{A}$ for loss aversion above which there exist internal solutions increases without bound as the number of states increases.

3.2.1 Adjusted Risk Premium

To understand which one of the two values in (5) is larger, we define the "adjusted risk premium":

$$\Delta \equiv \mu - \frac{\gamma\sigma^2}{2} - r_f. \tag{7}$$

It equals the "risk-adjusted expected return," $\mu - \frac{\gamma\sigma^2}{2}$, minus the riskless rate, r_f. The risk-adjusted expected return captures the trade-off between the expected return and the risk of the risky asset (adjusted for the agent's risk aversion). When $\Delta = 0$, the agent is indifferent between the risky asset and the riskless asset. When $\Delta > 0$, investing all wealth in the risky asset provides higher utility compared to investing in the riskless asset, and vice versa.

When $\Delta = 0$, the two values in (5) are the same ($\mathcal{C} = \mathcal{P}$), as demonstrated in Appendix A.7. If we interpret $\mathcal{C}$ and $\mathcal{P}$ as the prices of the "generalized call option" and "generalized put option" with a power-form payoff (with the power of $1-\gamma$), then $\Delta = 0$ leads to the put-call parity for the generalized options.[17] In this case, the lower bound of loss aversion is given by $\underline{A} = 1$. When $\Delta > 0$, we have $\mathcal{C} > \mathcal{P}$, and hence $\underline{A} = \frac{\mathcal{C}}{\mathcal{P}}$ (> 1). When $\Delta < 0$, we have $\underline{A} = \frac{\mathcal{P}}{\mathcal{C}}$ (> 1).

3.3 The Expected Utility

The expected utility function $U(\phi)$ satisfies

$$U = \frac{1}{1-\gamma}\Big\{\mathbb{E}\Big[(W_T - \theta)^{1-\gamma}\mathbf{1}_{\{W_T \geq \theta\}}\Big] - A\mathbb{E}\Big[(\theta - W_T)^{1-\gamma}\mathbf{1}_{\{W_T < \theta\}}\Big]\Big\}, \tag{8}$$

[17] Especially, when $\gamma = 0$, the call price becomes $\mathbb{E}[(\frac{R_T}{R_f} - 1)\mathbf{1}_{\{\frac{R_T}{R_f} \geq 1\}}] = c(1, 1, \nu, T, \sigma)e^{\nu T}$, where $\nu = \mu - r_f$, and $c(S_0, K, r, T, \sigma)$ is the Black-Scholes price of the European call option with stock price S_0, strike price K, interest rate r, maturity T, and volatility σ. Similarly, $\mathbb{E}[(1 - \frac{R_T}{R_f})\mathbf{1}_{\{\frac{R_T}{R_f} < 1\}}] = p(1, 1, \nu, T, \sigma)e^{\nu T}$, where $p(S_0, K, r, T, \sigma)$ is the European put option price. $\xi = 0$ is the same as the put-call parity. When $\gamma = 1$, (6) becomes the prices of binary call and put options.

where the terminal wealth W_T is given by (4). The expected utility consists of two components resulting from the gain domain ($W_T \geq \theta$) and the loss domain ($W_T < \theta$), respectively. The agent's choice is determined by the tradeoff of these two components.

We define three scenarios based on the dominance of the two components in (8). First, when $W_0 R_f < \theta$, the agent , starting with low initial wealth, experiences financial stressa situation we refer to as being "*under the water*." In this scenario, the loss component (i.e., the second term in (8)) dominates the expected utility. Conversely, when $W_0 R_f > \theta$, a situation referred to as being "*above the water*," the expected utility is primarily influenced by the gain component (i.e., the first term in (8)), with which the agent is risk averse. Finally, when $\theta = W_0 R_f$, the agent is "at the water."

The reference point plays a pivotal role in determining the scenario. To quantify its effect, we introduce the reference adjustment factor λ, defined as:

$$\lambda \equiv 1 - \frac{\theta}{W_0 R_f}. \tag{9}$$

This factor effectively measures the "depth of the water," encapsulating the impact of the reference point within the above-water and underwater scenarios, as demonstrated in Section 5.1. It is negative under the water, positive above the water, and zero at the water.

Lemma 2. *(Symmetry of the expected utility.)*

1. *If* $\Delta = 0$*, the expected utility* U *is symmetric about* $\phi = \frac{\lambda}{2}$*:* $U(\phi) = U(\lambda - \phi)$*.*

2. *If* $\Delta > 0$*,* $U(\phi) > U(\lambda - \phi)$ *for* $\phi > 0$*.*

3. *If* $\Delta < 0$*,* $U(\phi) > U(\lambda - \phi)$ *for* $\phi < 0$*.*

Lemma 2 shows that Δ controls the (a)symmetry of the expected utility as a function of the portfolio weight. The expected utility (8) equals the value of the option portfolio that is long one unit of a "generalized" call option and short A units "generalized" put option, where the call (put) option pays if wealth is higher (lower) than the reference point. When $\Delta = 0$, the prices of both options are symmetric with respect to the portfolio weight, and hence the expected utility is symmetric, independent of the loss aversion coefficient A. The symmetric expected utility is illustrated in Figure 2 center-right panels. In the underwater scenario (upper center-right panel), U has two local maximums, one being below the reference adjustment point $\phi^{*-} < \lambda$ and the other being positive $\phi^{*+} > 0$. According to Lemma 2, when $\Delta = 0$, the two local maximums are both global maximums and are symmetric about $\lambda/2$: $\phi^{*-} = \lambda - \phi^{*+}$. The short position $|\phi^{*-}|$ is larger than the long position $|\phi^{*+}|$ to offset the positive risk premium. In the above-water scenario (lower center-right panel), because U has a unique local maximum that occurs over $\phi \in [0, \lambda]$, Lemma 2 shows that when $\Delta = 0$, the optimal portfolio weight is given by $\phi^* = \frac{\lambda}{2}$ and is positive.

The expected utility U is not symmetric when $\Delta \neq 0$. The expected utility on the right of $\phi = \frac{\lambda}{2}$ is higher than that on the left if $\Delta > 0$, and the left part of U is higher otherwise.[18]

3.4 The Optimal Portfolio Weight

The following proposition summarizes the optimal portfolio weights.

Proposition 1. *(Optimal portfolio weight.) Assume* $A > \underline{A}$.

1. *When the agent is under the water* ($W_0 R_f < \theta$),
   (a) *for* $\Delta > 0$, $\phi^* \in (0, +\infty)$;
   (b) *for* $\Delta < 0$, $\phi^* \in (-\infty, \lambda)$;
   (c) *for* $\Delta = 0$, *there exist multiple optimal portfolio weights that are outside the range* $(\lambda, 0)$ *and are symmetric about* $\phi = \frac{\lambda}{2}$.

   In this scenario, the value function is negative.

2. *When the agent is above the water* ($W_0 R_f > \theta$),
   (a) *for* $\mu - r_f \geq \gamma\sigma^2$, *the optimal portfolio weight* ϕ^* *satisfies* $\phi^* \in [\lambda, +\infty)$;
   (b) *for* $0 < \mu - r_f < \gamma\sigma^2$, $\phi^* \in (0, \lambda)$; *particularly, (i)* $\phi^* \in (\lambda/2, \lambda)$ *if* $\Delta > 0$; *(ii)* $\phi^* = \lambda/2$ *if* $\Delta = 0$; *(iii)* $\phi^* \in (0, \lambda/2)$ *if* $\Delta < 0$;
   (c) *for* $\mu - r_f \leq 0$, $\phi^* \in (-\infty, 0]$.

   In this scenario, the value function is positive.

[18] Lemma 2 further implies symmetric optimal portfolio weights across assets. Consider two situations for problem (3): $\Delta = -\hat{\Delta}$. The optimal portfolio weights in the two situations satisfy $\phi^* = \lambda - \hat{\phi}^*$.

3. *When the agent is at the water ($W_0 R_f = \theta$), the optimal portfolio weight is $\phi^* = 0$, and the value function is zero.*

The loss aversion utility function with infinite loss aversion ($A = \infty$) reduces to HARA utility, and the optimal portfolio weight is summarized in the following corollary.

Corollary 1. *(The HARA benchmark.) Assume the agent has HARA utility ($A = \infty$).*

1. *Under the water ($W_0 R_f < \theta$), the HARA utility is not well-defined.*

2. *Above the water ($W_0 R_f > \theta$), the optimal portfolio weight ϕ^*_{hara} satisfies*

   (a) $\phi^*_{hara} = \lambda$ when $\mu - r_f > \gamma\sigma^2$.

   (b) $\phi^*_{hara} = \phi^*$ and $\phi^*_{hara} \in (0, \lambda)$ when $0 \leq \mu - r_f \leq \gamma\sigma^2$.

   (c) $\phi^*_{hara} = 0$ when $\mu - r_f < 0$.

3. *At the water ($W_0 R_f = \theta$), $\phi^*_{hara} = 0$.*

In particular, the HARA utility reduces to the CRRA utility when $\theta = 0$. In this case, when $W_0 > 0$, the optimal portfolio weight $\phi^{\circ *}_{hara} = 1$ for $\mu - r_f > \gamma\sigma^2$, $\phi^{\circ *}_{hara} \in (0, 1)$ for $0 \leq \mu - r_f \leq \gamma\sigma^2$, and $\phi^{\circ *}_{hara} = 0$ for $\mu - r_f < 0$; when $W_0 = 0$, $\phi^{\circ *}_{hara} = 0$; and when $W_0 < 0$, the CRRA utility is not well-defined. Here, the superscript " $\circ$" represents the case with a reference point of 0.

Corollary 1 shows that infinite portfolio weights can never be optimal under HARA. This is because the HARA utility imposes infinite penalty for losses, preventing the expected utility from approaching positive infinity. In contrast, the risk-averse agent with a low risk aversion coefficient tends to take infinite positions in the risky asset as shown in Lemma 1.

4 Properties of the Optimal Choices

Proposition 1 highlights significant differences in the optimal portfolios across the three scenarios: underwater, above water, and at the waterline. This section delves into the characteristics of the optimal choices in each scenario, with particular emphasis on the underwater case to explore the impact of risk-seeking behavior. Unless stated otherwise, we assume in this section that the bounded solution condition $A > \underline{A}$ holds.

4.1 Under the Water ($W_0R_f < \theta$)

In the underwater scenario, where the agent starts with low initial wealth, the expected utility is dominated by the loss component, leading to pronounced risk-seeking behavior. This behavior manifests in three key features: (1) a misalignment between the signs of the position and the risk premium, (2) unusually large risky positions, and (3) a "schizophrenia" behavior. These features are distinct and do not arise under risk-averse preferences.

4.1.1 The Sign of the Optimal Portfolio Weight

Proposition 1 demonstrates that in the underwater scenario, the sign of the optimal portfolio weight is determined by Δ, rather than the risk premium. The portfolio weight is positive when $\Delta > 0$ and negative when $\Delta < 0$. This contrasts with both risk-averse preferences commonly studied in the literature (e.g., HARA) and the above-water scenario under loss aversion, where in both cases, the sign of ϕ^* aligns with the sign of the risk premium.[19]

The misalignment between the signs of the optimal portfolio weight and the risk premium arises from the disconnect between local and global properties caused by risk-seeking behavior. For differentiable increasing utility functions, a small increase in the portfolio weight from zero always increases (decreases) the expected utility when the risk premium is positive (negative) (e.g., Arrow, 1971). That is,

$$sign\Big(\frac{\partial U}{\partial \phi}\Big|_{\phi=0}\Big) = sign(\mu - r_f). \tag{10}$$

This result holds true for risk aversion (i.e., concave) utility functions, as well as for loss aversion utility functions in both underwater and above-water scenarios ($W_0R_f \neq \theta$).

Under risk aversion utility functions, the uniform concavity implies that $U'(\phi) > 0$ if and only if $\phi < \phi^*$. This, together with (10), guarantees that the sign of ϕ^* is the same as the sign of the risk premium. However, the loss aversion utility function (1) is not uniformly concave, and its local and global properties are disconnected. In the underwater scenario, the sign of the optimal portfolio weight is not governed by the sign of risk premium but by the dominance between the generalized call and put options, which is dictated by the sign of Δ, as described in Lemma 2.

[19] This result is a direct implication from Lemma 2: when $\Delta > 0$, for any portfolio weight $\psi < \lambda$, there exists a portfolio weight $\phi = \lambda - \psi > 0$ such that $U(\phi) > U(\psi)$. In the underwater scenario, $\lambda < 0$. Therefore, the global maximum of U occurs at $\phi^* > 0$.

The signs of the optimal portfolio weights are summarized in the following corollary.

Corollary 2. *(The sign of the optimal portfolio weight.)*

1. *In the underwater scenario, the optimal portfolio weight is positive for* $\Delta > 0$ *and negative for* $\Delta < 0$*, and positive and negative portfolio weights can simultaneously be optimal for* $\Delta = 0$*.*

2. *In the above-water scenario, the sign of the optimal portfolio weight is the same as the sign of the risk premium* $\mu - r_f$*.*

Recall that $\Delta \equiv \mu - r_f - \frac{\gamma\sigma^2}{2}$. The sign of the optimal positions thus depends on both the return distributions and the curvature parameter γ. The loss-averse agent may even short a risky asset with a positive risk premium if the adjusted risk premium, Δ, is negative. In contrast, a HARA agent is always long such an asset (Corollary 1). This difference arises because a loss-averse agent under the water prioritizes achieving large returns to recover from losses, thereby placing less emphasis on risk. In this case, there is a "risk-return doubledown", driven by the global nature of risk-seeking behavior, rather than the typical risk-return tradeoff, which is a more localized phenomenon.

While these results are derived in the context of incomplete markets, Li et al. (2024) show that they also hold in complete markets, where a loss-averse agent under the water may short an asset with a positive risk premium or take a long position in an asset with a negative risk premium.

4.1.2 Large Risky Positions

The left panels of Figure 2 illustrate the expected utility as a function of portfolio weight in the underwater scenario. They demonstrate that the expected utility is bimodal, with two local maxima: one corresponding to a short position and the other to a long position in the risky asset. When the agent is underwater, starting with low initial wealth, moderate levels of risk exposure ($\phi \in [\lambda, 0]$) are suboptimal. This is because the resulting wealth remains below the reference point across all states, which the agent finds undesirable. To recover from losses, the agent takes on substantial risks, resulting in unusually large positions, either "stressed long" or "stressed short," in the risky asset. The position sizes in the underwater scenario (shown in the left panels of Figure 2) are significantly larger than those in the above-water scenario (the right panels). In addition, the position sizes in the underwater scenario are also much larger than those under HARA (Figure 3). These large risky positions in the underwater scenario align with individuals' tendency to engage in gambling behavior, even when the odds are unfavorable. Our results demonstrate that taking risk can be optimal under the expected utility framework.

This further suggests that when the agent is underwater, she consistently participates in the stock market, even if the stock has a zero risk premium. She opts to gamble, as it provides an opportunity to return to the gain domain. Notably, non-participation in the stock market ($\phi = 0$) is never an optimal strategy in the underwater scenario.

Moreover, when the agent is under the water, if she shorts the asset, she tends to short a large amount, lower than the reference adjustment factor $\lambda = 1 - \frac{\theta}{W_0 R_f}$. The larger the deviation of the riskless return scaled wealth from the reference point, the lower this factor is, leading to a large shorting position.

4.1.3 Schizophrenia

The bimodal expected utility further leads the agent to be schizophrenic. When $\Delta = 0$, the lower-middle left panel of Figure 2 shows that there are two optimal portfolio weights: one involves short selling, and the other involves leveraging. Both of these portfolio strategies lead to the same highest expected utility. The jump in the optimal portfolio weight occurs when Δ changes sign, causing a rapid shift in dominance between the generalized call and put options. As a result, even a small change in market conditions can trigger a dramatic shift in the optimal portfolio weight, prompting the agent to transition from aggressive short selling to substantial leveraging. This behavior is illustrated in the right panel of Figure 3 as the return distribution changes, and in the right panel of Figure 4 as preferences evolve.

The schizophrenic behavior does not occur under standard risk-averse preferences. Analyzing risk-seeking behavior entails both local and global considerations. The two local maxima are identified through the first-order conditions (FOCs), representing a local analysis. Determining the optimal portfolio weight, however, necessitates a global comparison of these maxima, with the schizophrenia arising from this comparison. In contrast, under risk aversion preferences, the FOCs alone are sufficient to determine optimality.

This schizophrenia behavior can align with empirical observations. For example, Coval and Shumway (2005) find that following morning losses, professional market makers are far more likely to take on additional afternoon risk and trade (either buy or sell) more aggressively.

In our model, market incompleteness limits the impact of risk-seeking behavior. Li et al. (2024) show that in complete markets, an agent who is underwater allocates her wealth to be positive in all but one state, while taking a negative position in the most expensive state. This amplifies the schizophrenia behavior, which becomes more pronounced with the presence of multiple local maxima in the expected utility function. Complete and incomplete markets display the largest differences in the optimal portfolio weights when the agent is underwater but lead to similar properties when the agent is above-water.

4.1.4 Isolating Risk Seeking in Loss Aversion Preferences

The above features of optimal choices in the underwater scenario are fundamentally driven by risk seeking. To illustrate this, we compare it to a special case with $\gamma = 0$, which isolates the effects of loss aversion without introducing risk-seeking preferences. In this case, the utility function becomes bilinear with a kink, remaining concave with $A > 1$. Consequently, the bilinear utility function leads to globally risk-averse but locally risk-neutral behavior. Unlike the case with $\gamma > 0$, it does not produce risk-seeking tendencies. However, the other two utility features—reference dependence and loss aversion—are still present. This special case with $\gamma = 0$ has been widely studied in the literature (e.g., Barberis et al., 2001).

When $\gamma = 0$, the generalized options (6) become vanilla options. The following corollary summarizes the results.

Corollary 3. *(Optimal portfolio weight under bilinear utility function $\gamma = 0$.)*

1. *When $\mu - r_f > 0$, there is a unique optimal portfolio weight ϕ^* such that*

$$\Phi(d_1) - R_f e^{-\mu T}\Phi(d_2) = A\big[R_f e^{-\mu T}\Phi(-d_2) - \Phi(-d_1)\big], \tag{11}$$

where $\Phi(\cdot)$ is the standard normal CDF, and

$$d_1(\phi) = \frac{\ln(\frac{1}{R_f(1-\lambda/\phi)}) + (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \qquad d_2(\phi) = d_1(\phi) - \sigma\sqrt{T}. \tag{12}$$

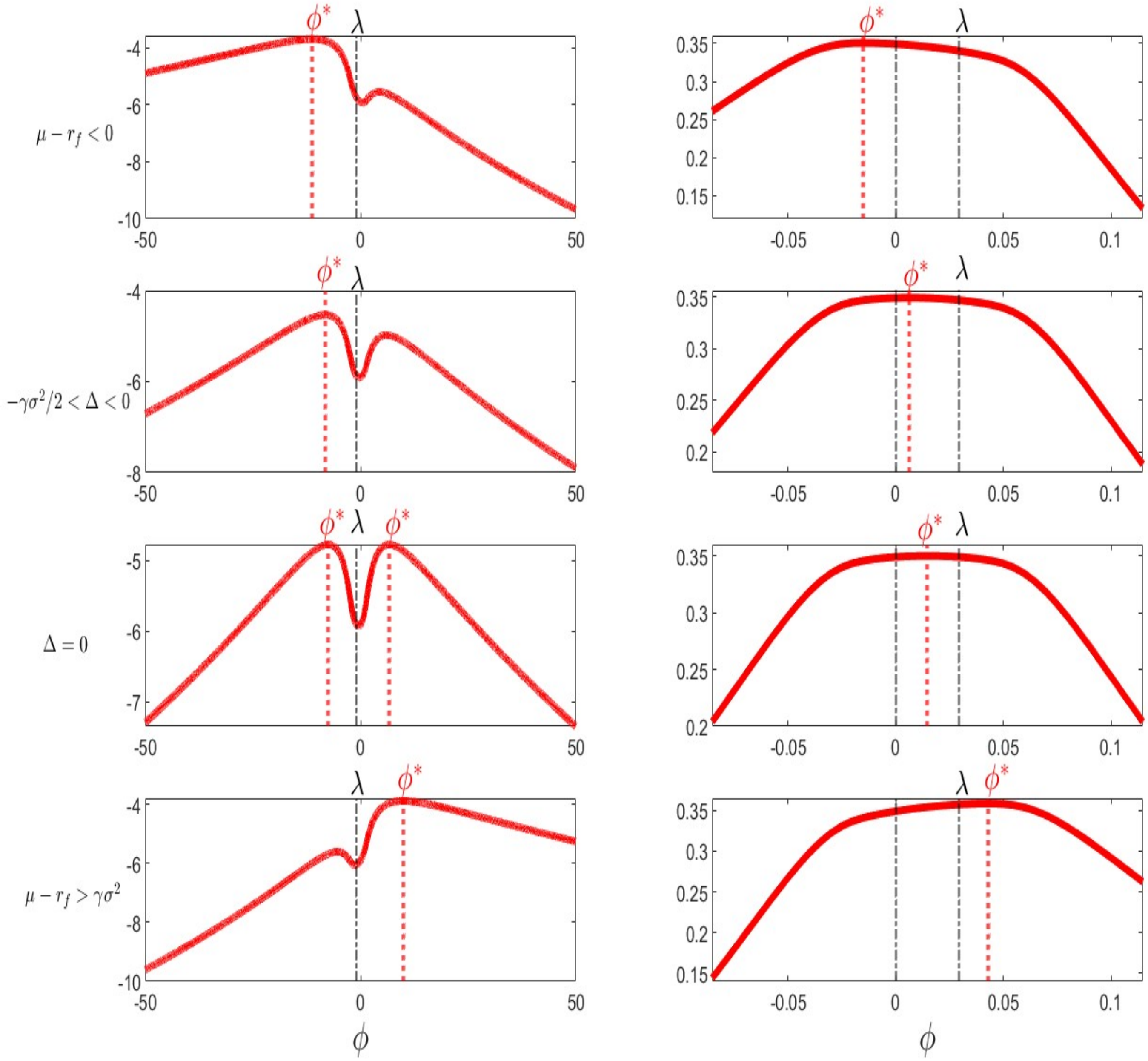

Figure 2: The figure plots the expected utility function U against the portfolio weight ϕ in the underwater scenario (the left panels) and the above-water scenario (the right panels). Here, $A = 3$ $(> \underline{A})$, $\gamma = 0.5$, $W_0 = 1$, $T = 1$, $r_f = 0.03$, $\sigma = 0.3$, and $\theta = 2$ in the left panels and $\theta = 1$ in the right panels. We set the risk premium $\mu - r_f$ equal to -0.03 in the top panels (such that $\mu - r_f < 0$), 0.01 in upper-middle panels ($-\gamma\sigma^2/2 < \Delta < 0$), $\gamma\sigma^2/2$ in lower-middle panels ($\Delta = 0$), and 0.07 in bottom panels ($\mu - r_f > \gamma\sigma^2$).

2. When $\mu - r_f < 0$, there is a unique optimal portfolio weight such that

$$A\big[\Phi(d_1) - R_f e^{-\mu T}\Phi(d_2)\big] = R_f e^{-\mu T}\Phi(-d_2) - \Phi(-d_1). \tag{13}$$

3. When $\mu - r_f = 0$, all values between 0 and λ (inclusive) are optimal portfolio weights.

Several observations follow Corollary 3. First, the misalignment between the signs of the position and the risk premium does not occur when $\gamma = 0$. In this case, the utility function is concave. Consequently, when $\mu - r_f \neq 0$, the sign of the portfolio weight always aligns with the sign of the risk premium, regardless of whether the agent is underwater or above water. This is in stark contrast to the case with $\gamma > 0$, where risk-seeking behavior can lead to a misalignment.

Second, the schizophrenia behavior scenario also does not occur, unlike the risk seeking case with $\gamma > 0$. The expected utility function is concave, leading to a unique local maximum in the underwater scenario.[20]

Third, when $\gamma = 0$, the optimal portfolio weights are qualitatively similar across the underwater and above-water scenarios. This stands in stark contrast to the case with $\gamma > 0$, where the optimal portfolio weights exhibit distinct characteristics between the two scenarios. Numerical simulations (not shown here) indicate that the primary difference for $\gamma = 0$ lies in the size of the risky positions: the positions tend to be larger in the underwater scenario and relatively smaller above water. The tendency toward large positions is a common prediction of both risk-seeking and risk-neutral behaviors.

Additionally, in the knife-edge case $\mu - r_f = 0$, any portfolio weights within the range between 0 and λ (inclusive) is considered optimal. For example, in the underwater scenario, if the portfolio weight satisfies $\phi \in [\lambda, 0]$, the agent's wealth remains in the loss domain across all states. In this situation, the agent becomes effectively risk-neutral and is indifferent among all portfolio weights within this range. These weights are optimal because, outside this interval, the agent's terminal wealth spans both the loss and gain domains depending on the state, leading to reduced expected utility due to her global risk-averse tendencies.

[20]The expected utility is a linear function between 0 and λ, of which the slope is the same as the sign of the risk premium, and it is strictly concave outside this interval.

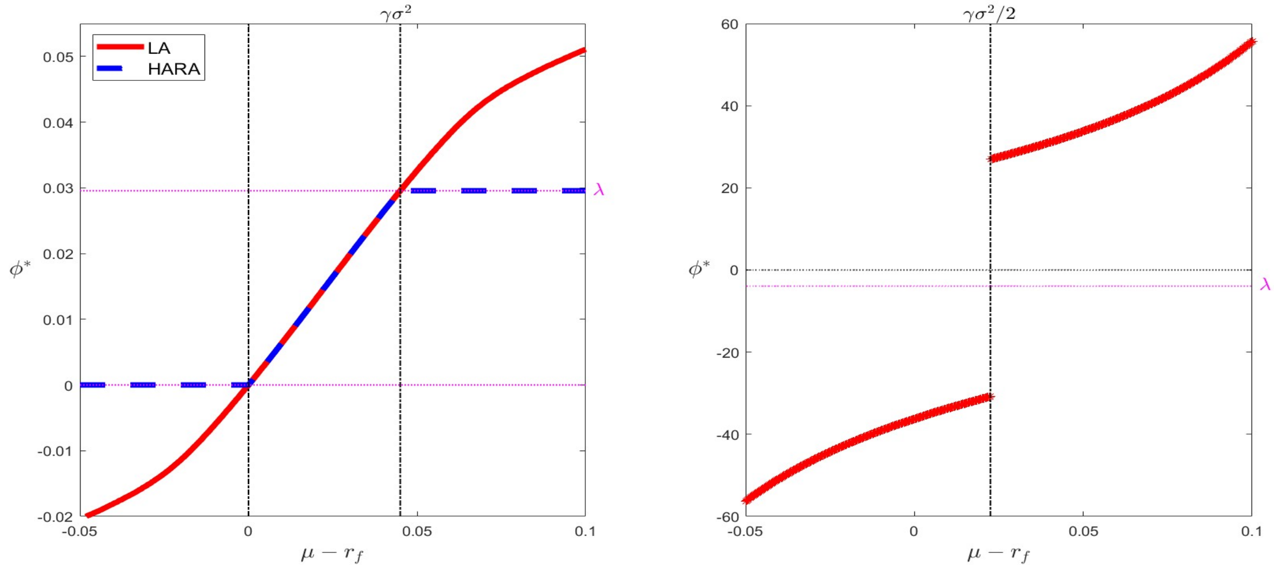

Figure 3: The figure illustrates the impacts of the risk premium on the optimal portfolio weight. The left panel compares the optimal portfolio weight under the loss aversion (LA) utility function and the HARA in the above-water scenario, and the right panel plots the optimal portfolio weight under loss aversion in the underwater scenario. Here, $A = 3$, $\gamma = 0.5$, $\theta = 1$ in the left panel and $\theta = 5$ in the right panel, $W_0 = 1$, $T = 1$, $r_f = 0.03$, $\mu \in [-0.02, 0.13]$, and $\sigma = 0.3$.

4.2 Above the Water ($W_0 R_f > \theta$)

In the above-water scenario, where the agent begins with high initial wealth, the expected utility is primarily influenced by the gain component. This results in behavior closely resembling standard risk-averse preferences. This scenario is predominantly studied in the loss aversion literature, e.g., Benartzi and Thaler (1995) and Berkelaar et al. (2004).

The right panels of Figure 2 illustrate the expected utility in the above-water scenario. The expected utility has a unique local (also global) maximum, which occurs for small risky positions. The agent tends to maintain small positions and low volatility in her wealth. On the one hand, the loss-averse agent above the water behaves similarly to standard risk-averse agents, who typically take small risky positions under plausible parameters. On the other hand, the agent needs to allocate a fixed amount of her wealth to the cash account to offset the reference point, with any additional wealth then allocated between the assets. This further lowers her risky position when she is above the water. In contrast, the agent takes large positions under the water, which lead to a much more dispersed wealth distribution.

In addition, the sign of the optimal portfolio weights is the same as that of the risk premium, a common result under risk aversion preferences. Therefore, all the three key features in the underwater scenario—misalignment between the signs of position and risk premium, large risky positions, and schizophrenia—disappear in the above-water scenario.

Moreover, by comparing Proposition 1 and Corollary 1, it becomes evident that, in the above-water scenario, the loss-averse agent exhibits behavior similar to that of a HARA agent, albeit in a more aggressive manner. When the risk premium is positive but below $\gamma\sigma^2$—the minimum level of risk premium at which a CRRA agent would allocate all her wealth to the risky asset—the optimal portfolio weight under loss aversion is identical to that under the HARA utility function. In this case, the loss aversion coefficient A does not affect the optimal portfolio weight. Outside this interval (i.e., $\mu - r_f < 0$ or $\mu - r_f > \gamma\sigma^2$), the loss-averse agent trades more aggressively than the HARA agent, who never shorts or leverages due to the infinite marginal utility at $W_T = \theta$.[21] These findings for the above-water scenario are illustrated in the left panel of Figure 3.

4.3 At the Water ($\theta = W_0 R_f$)

When $\theta = W_0 R_f$, the optimal portfolio weight satisfies

$$\phi^* = \begin{cases} 0, & \text{if} \quad A > \underline{A}; \\ \pm\infty, & \text{if} \quad A < \underline{A}; \\ \forall \phi, & \text{if} \quad A = \underline{A}. \end{cases}$$

Under the boundedness solution condition ($A > \underline{A}$), the agent never invests in the stock. This result has been documented in the literature, e.g., He and Zhou (2011), and is opposite to that under the water, in which non-participation in the stock market is never optimal. In fact, first-order risk aversion (Segal and Spivak, 1990) applies at the reference point of the loss aversion utility function, causing the agent to be reluctant to take on small risks when

[21] In incomplete markets, the loss-averse agent must suffer losses in some states when $\mu - r_f < 0$ or $\mu - r_f > \gamma\sigma^2$, causing her to be risk seeking. When the markets are complete, the optimal wealth is always in the gain domain when the scenario is above the water, and thus the optimal portfolios under the loss aversion and HARA are always identical in the above-water scenario (Li et al. 2024).

her terminal wealth is at the reference point.[22]

5 Comparative Statics

In this section, we study the effects of parameters, including both preference parameters and assets' return parameters, on the expected utility and the optimal portfolio weight.

5.1 Effects of the Reference Point

The reference point θ is one of the most important features of loss aversion. The choice of it is a key challenge in the application of prospect theory (Barberis, 2013), and the literature has proposed different choices for it. Our paper takes the reference point as given but provide a general analysis of its effects for any specified reference level. We have shown that the most significant effect of the reference point is determining the watermark, above and under which the optimal portfolio weights are distinctly different (Proposition 1).

In this subsection, we further analyze the impact of the reference point within each scenario. This impact is assessed by comparing it to the homogeneous case with a reference point of $\theta = 0$. The reference adjustment factor λ, as defined in (9), serves as a sufficient statistic for capturing this effect.

With a reference point of 0, the expected utility in (8) becomes

$$U^\circ = \begin{cases} \frac{(W_0^\circ)^{1-\gamma}}{1-\gamma}\mathbb{E}\big[(R_W^\circ)^{1-\gamma}\mathbf{1}_{\{R_W^\circ \geq 0\}} \; - \; A(-R_W^\circ)^{1-\gamma}\mathbf{1}_{\{R_W^\circ < 0\}}\big], & \text{if } W_0^\circ > 0; \\ 0, & \text{if } W_0^\circ = 0; \\ \frac{(-W_0^\circ)^{1-\gamma}}{1-\gamma}\mathbb{E}\big[(-R_W^\circ)^{1-\gamma}\mathbf{1}_{\{R_W^\circ \leq 0\}} \; - \; A(R_W^\circ)^{1-\gamma}\mathbf{1}_{\{R_W^\circ > 0\}}\big], & \text{if } W_0^\circ < 0, \end{cases} \tag{14}$$

where $R_W^\circ = R_f + \phi^\circ(R_T - R_f)$ is the gross return of wealth, and the superscript $\circ$ represents the case with a reference point of 0. With a zero reference point $\theta = 0$, the utility function is homogenous in its argument. Consequently, for two optimization problems with initial

[22]The non-participation can be also understood from the finding in Bowman, Minehart and Rabin (1999) and Rabin (2000) that loss aversion leads to the rejection of any "slightly-better-than-fair bet", which can be accepted by a risk-averse agent. However, it seems that Bowman et al. (1999) and Rabin (2000) implicitly assume that the marginal utility for standard expected-utility theory has to be finite, which is not the case for the HARA utility at the reference point.

wealth of the same sign, the optimal portfolio weights are identical. However, if the initial wealth has the opposite signs, then the optimal portfolio weights are starkly different.[23]

Proposition 2. *(Reference adjustment.) Denote by ϕ^* and W^* the optimal portfolio weight and the optimal wealth, respectively, under the loss aversion utility with a reference point θ and initial wealth W_0.*

1. *Under the water,*
$$\phi^* = \lambda \hat{\phi}^{\circ *}, \qquad W_T^* = \theta - \lambda \hat{W}_T^{\circ *}, \tag{15}$$
where $\hat{\phi}^{\circ *}$ and $\hat{W}_T^{\circ *}$ are the optimal portfolio weight and the optimal wealth, respectively, under the loss aversion utility with a reference point 0 and initial wealth $\hat{W}_0 = -W_0$.

2. *Above the water,*
$$\phi^* = \lambda \phi^{\circ *}, \qquad W_T^* = \theta + \lambda W_T^{\circ *}, \tag{16}$$
where $\phi^{\circ *}$ and $W_T^{\circ *}$ are the optimal portfolio weight and the optimal wealth, respectively, under the loss aversion utility with a reference point 0 and initial wealth W_0.

Proposition 2 describes the relationship between the optimal portfolios under the loss aversion utility functions with reference points θ and 0. For $\lambda \neq 0$, there is a one-to-one correspondence between the optimal portfolio weights under the two reference points. Intuitively, by investing θR_f^{-1} in the riskless asset and the remaining wealth $W_0 - \theta R_f^{-1}$ $(= \lambda W_0)$ in the optimal portfolio weights under the utility with $\theta = 0$, the resultant portfolio is optimal under the original loss aversion utility with reference point θ. Therefore, the optimization problem with initial wealth W_0 and reference point θ is equivalent to the optimization problem with initial wealth λW_0 and reference point 0.

The relationship between the optimal portfolio weights as in (15) and (16) show that λ measures the effect of the depth of the water. For example, if the agent is under the water, a further decrease in the agent's wealth exacerbates her financial situation, causing the agent to take larger (either long or short) risky positions. The above results also apply to the HARA utility as shown in Appendix B.

[23]Especially, when A is close to 1 or when A is sufficiently large, the expected utility functions with opposite initial wealth tend to have the opposite monotonicity, and hence the portfolio weight that leads to a local maximum (minimum) of the expected utility with positive initial wealth tends to generate a local minimum (maximum) of the expected utility with negative initial wealth.

Proposition 2 further shows that within each scenario, the effects of the reference point θ on the optimal portfolio weight and the optimal wealth are completely captured by the reference adjustment factor λ. The larger the deviation of $W_0 R_f$ from θ, the more aggressively the agent trades. For example, when wealth is above the reference point, a higher reference point leads to higher risk aversion, which lowers the optimal portfolio weight.

Corollary 4. *(Effects of the reference point.) Consider two optimization problems with different reference points, θ and $\hat{\theta}$, while keeping all other parameters the same. If λ ($\equiv 1 - \frac{\theta}{W_0 R_f}$) and $\hat{\lambda}$ ($\equiv 1 - \frac{\hat{\theta}}{W_0 R_f}$) have the same sign, the optimal portfolio weights and the optimal wealth under the two utility functions satisfy*

$$\phi^* = \frac{\lambda}{\hat{\lambda}} \hat{\phi}^*, \qquad W_T^* - \theta = \frac{\lambda}{\hat{\lambda}} (\hat{W}_T^* - \hat{\theta}). \tag{17}$$

Corollary 4 shows that for two portfolio problems with different reference points, within the same under/above-water scenario), an increase in the reference point causes the agent to allocate more wealth to the cash account to adjust for the higher reference point. This adjustment reduces her risky position when she is above the water, but it increases her risky position when she is underwater since a higher reference point pushes her further into negative territory. If λ and $\hat{\lambda}$ have different signs (i.e., in different scenarios), there is no one-to-one mapping between the optimal portfolio weights, and the optimal portfolios exhibit starkly different properties.

In summary, the above results show that, within a scenario, the effect of the reference point θ is no more than a change of variable, without affecting the sign of ϕ^*.

5.1.1 Effects of θ on Risk Aversion and the Equity Premium Puzzle

For the loss aversion utility function (1), when $W_T - \theta \neq 0$, the relative risk aversion of is given by $-W_T \frac{u''(W_T)}{u'(W_T)} = \gamma \frac{W_T}{W_T - \theta}$. It follows that a higher reference point of the loss aversion utility function causes the agent to be more risk averse in the gain domain $W_T - \theta > 0$, as more wealth needs to be invested in the riskless asset to offset the reference point. This effect of the reference point is identical to that for the HARA utility that features higher risk aversion than the corresponding CRRA utility. However, in the loss domain $W_T - \theta < 0$, the agent with a higher reference point tends to be more risk seeking.

Loss aversion has been used to explain the equity premium puzzle (e.g., Benartzi and Thaler, 1995; Barberis et al., 2001). We show that a higher reference point causes the agent

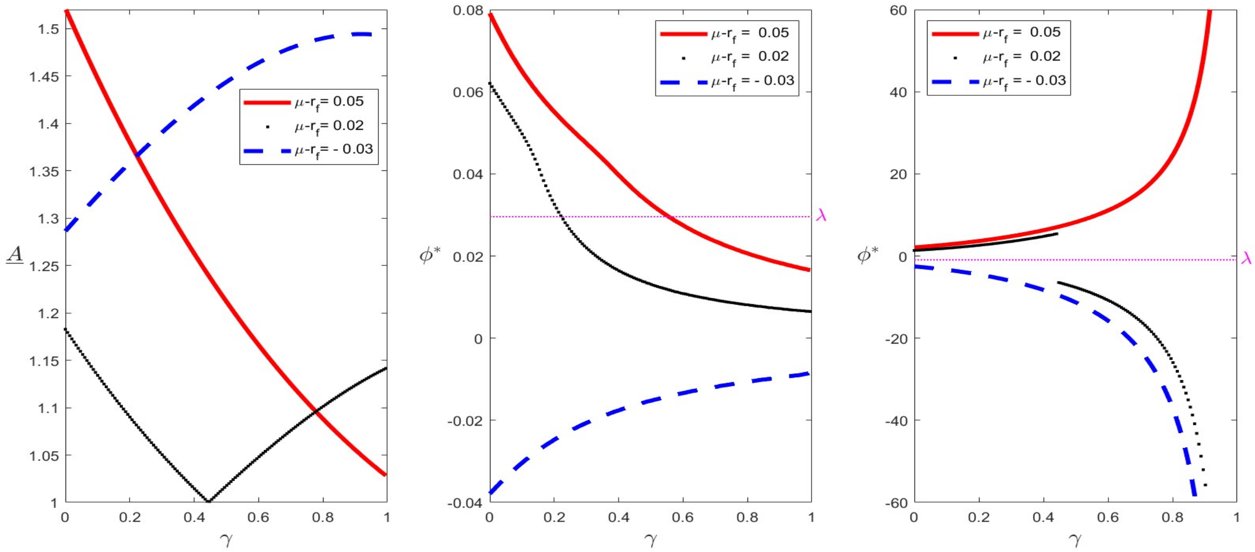

Figure 4: The left panel plots $\underline{A}$ against γ. The middle and right panels plot the optimal portfolio weight ϕ^* against γ for the case $\theta < W_0R_f$ (with $\theta = 1$) and the case $\theta > W_0R_f$ (with $\theta = 2$), respectively. The other parameters are given by $A = 3$ (higher than $\underline{A}$), $W_0 = 1$, $T = 1$, $r_f = 0.03$, and $\sigma = 0.3$.

to be more risk averse in the gain domain. As a result, the agent tends to require a higher rate of return of the risky asset, helping resolve the equity premium puzzle. However, our results on the relationship between loss aversion and HARA suggest that risk seeking in loss aversion preferences tends to amplify the puzzle: a loss-averse agent requires a smaller risk premium than the corresponding HARA agent.

5.2 Effects of the Curvature Parameter

Figure 4 plots the optimal portfolio weight ϕ^* against γ above (in the middle panel) and under (in the right panel) the water. Above the water, the absolute value of the optimal portfolio weight decreases with γ. Because the gain term dominates the expected utility, as γ decreases, the agent becomes less risk averse and holds more risky portfolios. When $0 < (\mu - r_f)/\sigma^2 \leq \gamma \leq 1$ (the red solid line and black dotted line below λ), the optimal portfolio weight under loss aversion is the same as that under the HARA utility. In this case, γ measures only risk aversion since the optimal portfolio is always in the gain domain. When $0 < \gamma < (\mu - r_f)/\sigma^2$ (the red solid line and black dotted line above λ), the loss-averse agent trades more aggressively than the HARA agent who has an optimal portfolio weight λ.

In the underwater scenario, Figure 4 right panel shows that the absolute value of the optimal portfolio weight increases with γ. This result is at odds with traditional risk-averse preferences. In fact, because the loss component dominates the expected utility, as γ increases, the agent becomes more risk seeking and hence takes larger positions (either long or short) in the risky asset.

In the right panel, the black dotted line shows that as γ increases, there is a jump in the optimal portfolio weight. In the underwater scenario, the expected utility has two local maximums, one with positive and one with negative portfolio weights, and the global maximum is the greater of them. Figure 4 right panel shows that the optimal portfolio weight is positive when γ is small and negative when γ is large, and the optimal portfolio weight ϕ^* jumps at a threshold ($\gamma \approx 0.44$), at which the two local maximums are the same, leading to discontinuity in the optimal portfolio weight (see Corollary 2). However, when the risk premium is sufficiently large (or negative) such that $\Delta > 0$ ($\Delta < 0$) for all γ, there is no jump in the optimal portfolio weight.

5.3 Effects of the Loss Aversion Coefficient

In the loss aversion utility function (1), the loss aversion coefficient A directly determines the local loss aversion around the reference point. Lemma 1 further shows that it also determines the global properties. When A is sufficiently low, the penalty for losses is small, and the agent tends to take infinite positions in the risky asset. As A increases, the penalty for losses increases, and the agent takes smaller risky positions (either long or short). In the extreme case when $A \to +\infty$, the loss aversion utility function becomes the HARA.

5.4 Effects of Return Parameters

Return parameters significantly affect the decision-making under loss aversion. First, the boundedness of the optimal portfolio weight depends on the return distributions of the assets, as stated in Lemma 1. Second, return parameters determine Δ in (7) that determines the sign of the positions, and a small change in the market conditions can trigger the schizophrenia.

We first examine the effects of the risk premium. Above the water, the optimal portfolio weight always increases with the risk premium as shown in the left panel of Figure 3. Under the water (the right panel of Figure 3), the optimal portfolio weight is positive for $\Delta > 0$ and negative for $\Delta < 0$, and a small change in the market condition (around $\Delta = 0$) leads to a big jump in the optimal portfolio weight. The agent in this case is forced to take large risky positions, either leverage or shorting the risky asset. Importantly, the right panel of Figure 3 shows that the agent shorts an asset with a zero or even positive risk premium: $0 < \mu - r_f < \gamma\sigma^2/2$.

The effects of return volatility are twofold. First, because the adjusted risk premium Δ is negatively related with return volatility, a rise in volatility can turn a positive Δ negative. This shift can trigger the schizophrenic behavior, causing the agent to transition from substantial leveraging to aggressive short selling.

Second, the expected utility (in absolute value) is inversely related to return volatility with each scenario (see (A.1) and (A.2) in Appendix A.3). As a result, under the boundedness condition (in Lemma 1), the magnitude of optimal positions decreases with increasing return volatility, whether in the underwater or above-water scenario. Intuitively, higher volatility, holding all else equal, increases risk exposure and potential losses, which the agent seeks to avoid. Consequently, the agent reduces the size of her position in the risky asset.[24]

Finally, we discuss the effects of the riskless rate. The most significant effect of the riskless rate is affecting the watermark. The agent tends to be risk averse in the periods with low interest rates; however, she tends to be risk seeking in high interest rate periods. Notably, the return of the riskless asset determines the three scenarios, but the return distribution of the risky asset does not.

We further investigate the impact of the riskless rate within a given scenario. This effect, a direct consequence of Proposition 2, is summarized in the following corollary.

Corollary 5. *(Effects of the riskless rate and initial wealth.) Consider the optimal portfolio choice problems under the loss aversion utility function with the riskless rate and initial wealth* (r_f, W_0) *and* $(\hat{r}_f, \hat{W}_0)$*, respectively. If* λ $(\equiv 1 - \frac{\theta}{W_0 R_f})$ *and* $\hat{\lambda}$ $(\equiv 1 - \frac{\theta}{\hat{W}_0 \hat{R}_f})$ *have the same sign, the optimal portfolio weights and the optimal wealth for the two optimization problems satisfy*

$$\phi^* = \frac{\lambda}{\hat{\lambda}}\hat{\phi}^*, \qquad W_T^* - \theta = \frac{\lambda W_0}{\hat{\lambda}\hat{W}_0}(\hat{W}_T^* - \theta). \tag{18}$$

[24] Additionally, under the water, the agent is likely to short an asset with sufficiently large return volatility (due to a negative Δ), while a HARA agent holds virtually no position in this asset.

Consider positive initial wealth. Corollary 5 shows that in the above-water scenario with a positive reference adjustment factor, the optimal portfolio weight increases with the riskless rate. Moreover, in this scenario, the optimal portfolio weight is an increasing and concave function of the initial wealth, which is consistent with the household evidence documented in Calvet and Sodini (2014). This result is due to the positive reference point. However, in the underwater scenario with a negative reference adjustment factor, the optimal portfolio weight decreases with the riskless rate and the initial wealth.

6 Conclusion

Humans have moments when they are risk-seeking. This is a significant psychological attribute. However, economics predominantly focuses on one psychological attribute, namely, risk aversion. The formal modelling of risk aversion is elegant and tractable and has been well-studied. On the contrary, risk seeking has received much less attentions and has been largely overlooked.

This paper conducts a formal analysis of the implications of risk seeking. We adopt the loss aversion utility function—a fundamental component of prospect theory—that provides a parsimonious but realistic framework for risk seeking. We show that the agent takes large risky positions, swings between sizable long and short positions, and shorts assets with positive risk premia. These results are due to risk seeking. However, they are contradict with risk-averse behaviors.

Our results suggest that risk seeking deserves a permanent place in economic analysis, since it is ubiquitous and significantly affects decision-making.[25] It generates implications that are markedly different from those under risk aversion but consistent with human behaviors. Understanding its implications is important for individuals, firms, and even governments to deal with stressed situations.

[25] Our paper echoes the survey by Barberis (2013), which observes that reference dependence embedded in loss aversion preferences would likely find a permanent place in economic analysis.

A Proofs

A.1 Proof of Lemma 1

In the Appendix, we provide general proofs using the value of the holdings of the risky asset x, instead of the portfolio weight ϕ, as the former also applies when the initial wealth W_0 is zero or negative. When initial wealth is positive, it satisfies $x = W_0\phi$.

We are interested in large x behavior. For $x > 0$, the expected utility U is given by

$$\begin{aligned} U =& \frac{(xR_f)^{1-\gamma}}{1-\gamma}\mathbb{E}\Bigg[\left(\frac{R_T}{R_f} - 1 + \frac{W_0R_f - \theta}{xR_f}\right)^{1-\gamma}\mathbf{1}_{\{\frac{R_T}{R_f}-1+\frac{W_0R_f-\theta}{xR_f}\geq 0\}} \\ &- A\left(1 - \frac{R_T}{R_f} - \frac{W_0R_f-\theta}{xR_f}\right)^{1-\gamma}\mathbf{1}_{\{\frac{R_T}{R_f}-1+\frac{W_0R_f-\theta}{xR_f}<0\}}\Bigg]. \end{aligned}$$

When $x \to +\infty$, $U \to \frac{(xR_f)^{1-\gamma}}{1-\gamma}(\mathcal{C} - A\mathcal{P})$, which is bounded from above if $\mathcal{C} < A\mathcal{P}$, where $\mathcal{C}$ and $\mathcal{P}$ are given by (6). For $x < 0$,

$$\begin{aligned} U =& \frac{[(-x)R_f]^{1-\gamma}}{1-\gamma}\mathbb{E}\Bigg[\left(1 - \frac{R_T}{R_f} + \frac{W_0R_f - \theta}{-xR_f}\right)^{1-\gamma}\mathbf{1}_{\{1-\frac{R_T}{R_f}+\frac{W_0R_f-\theta}{-xR_f}\geq 0\}} \\ &- A\left(\frac{R_T}{R_f} - 1 - \frac{W_0R_f-\theta}{-xR_f}\right)^{1-\gamma}\mathbf{1}_{\{1-\frac{R_T}{R_f}+\frac{W_0R_f-\theta}{-xR_f}<0\}}\Bigg]. \end{aligned}$$

When $x \to -\infty$, $U \to \frac{[(-x)R_f]^{1-\gamma}}{1-\gamma}(\mathcal{P} - A\mathcal{C})$, which is bounded from above if $\mathcal{P} < A\mathcal{C}$. Therefore, the optimal portfolio weight is bounded when $A > \underline{A}$ and unbounded $A < \underline{A}$, where $\underline{A}$ is given by (5).

Suppose $A = \underline{A}$. If $W_0R_f < \theta$, $U < 0$, and U tends to either 0 or $-\infty$ as $x \to \pm\infty$; thus, the optimal portfolio weight is either positive infinity or negative infinity. If $W_0R_f > \theta$, $U > 0$, and U tends to either 0 or $-\infty$ as $x \to \pm\infty$. If $W_0R_f = \theta$, $U \equiv 0$.

A.2 Proof of Lemma 2

By defining $\Lambda \equiv W_0R_f - \theta$, we rewrite the expected utility $U(x)$ as

$$U(x) = \frac{R_f^{1-\gamma}}{1-\gamma}\left\{\mathbb{E}\Big[\Big(x(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{x(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}\geq 0\}} - A\Big(-x(\frac{R_T}{R_f}-1)-\frac{\Lambda}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{x(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}<0\}}\Big]\right\},$$

where

$$\Big(x(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}\Big)^{1-\gamma} =\Big(x(e^{(\mu-\frac{\sigma^2}{2}-r_f)T+\sigma\sqrt{T}\epsilon}-1)+\frac{\Lambda}{R_f}\Big)^{1-\gamma}$$

$$=\Big(x(e^{[\Delta-\frac{\sigma^2}{2}(1-\gamma)]T+\sigma\sqrt{T}\epsilon}-1)+\frac{\Lambda}{R_f}\Big)^{1-\gamma}$$

$$=e^{-\frac{\sigma^2}{2}(1-\gamma)^2T+\sigma(1-\gamma)\sqrt{T}\epsilon}\Big(xe^{\Delta}-(x-\frac{\Lambda}{R_f})e^{\frac{\sigma^2}{2}(1-\gamma)T-\sigma\sqrt{T}\epsilon}\Big)^{1-\gamma}.$$

Consider a new measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative: $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}=e^{-\frac{\sigma^2}{2}(1-\gamma)^2T+\sigma(1-\gamma)\sqrt{T}\epsilon}$. Under $\tilde{\mathbb{P}}$, $\tilde{\epsilon}=\epsilon-\sigma(1-\gamma)\sqrt{T}$ is a standard normal random variable. Then

$$\mathbb{E}\Big[\Big(x(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{x(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}\geq 0\}}-A\Big(-x(\frac{R_T}{R_f}-1)-\frac{\Lambda}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{x(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}<0\}}\Big]$$

$$=\tilde{\mathbb{E}}\Big[\Big(xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T-\sigma\sqrt{T}\tilde{\epsilon}}\Big)^{1-\gamma}\mathbf{1}_{\{xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T-\sigma\sqrt{T}\tilde{\epsilon}}\geq 0\}}$$
$$-A\Big(-xe^{\Delta T}+(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T-\sigma\sqrt{T}\tilde{\epsilon}}\Big)^{1-\gamma}\mathbf{1}_{\{xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T-\sigma\sqrt{T}\tilde{\epsilon}}<0\}}\Big]$$

$$=\tilde{\mathbb{E}}\Big[\Big(xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\tilde{\epsilon}}\Big)^{1-\gamma}\mathbf{1}_{\{xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\tilde{\epsilon}}\geq 0\}}$$
$$-A\Big(-xe^{\Delta T}+(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\tilde{\epsilon}}\Big)^{1-\gamma}\mathbf{1}_{\{xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\tilde{\epsilon}}<0\}}\Big]$$

$$=\mathbb{E}\Big[\Big(xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\epsilon}\Big)^{1-\gamma}\mathbf{1}_{\{xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\epsilon}\geq 0\}}$$
$$-A\Big(-xe^{\Delta T}+(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\epsilon}\Big)^{1-\gamma}\mathbf{1}_{\{xe^{\Delta T}-(x-\frac{\Lambda}{R_f})e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\epsilon}<0\}}\Big]$$

$$=e^{(1-\gamma)\Delta T}\Big\{\mathbb{E}\Big[\Big((\frac{\Lambda}{R_f}-x)(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}+x(e^{2\Delta T}-1)\Big)^{1-\gamma}\mathbf{1}_{\{(\frac{\Lambda}{R_f}-x)(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}+x(e^{2\Delta T}-1)\geq 0\}}$$
$$-A\Big(-(\frac{\Lambda}{R_f}-x)(\frac{R_T}{R_f}-1)-\frac{\Lambda}{R_f}-x(e^{2\Delta T}-1)\Big)^{1-\gamma}\mathbf{1}_{\{(\frac{\Lambda}{R_f}-x)(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}+x(e^{2\Delta T}-1)<0\}}\Big]\Big\},$$

which is greater (less) than

$$\mathbb{E}\Big[\Big((\frac{\Lambda}{R_f}-x)(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{(\frac{\Lambda}{R_f}-x)(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}\geq 0\}}-A\Big((x-\frac{\Lambda}{R_f})(\frac{R_T}{R_f}-1)-\frac{\Lambda}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{(\frac{\Lambda}{R_f}-x)(\frac{R_T}{R_f}-1)+\frac{\Lambda}{R_f}<0\}}\Big]$$

if $\Delta>0$ and $x>0$ ($\Delta<0$ and $x<0$). The two values are equal if $\Delta=0$. Note that the last equation is the expected utility function with stock holdings of $\frac{\Lambda}{R_f}-x$. Therefore, if $\Delta=0$, $U(x)=U(\frac{\Lambda}{R_f}-x)$.

A.3 Proof of Proposition 1

The loss aversion preference is inherently global in nature, determining the optimal portfolio weight necessitates a comprehensive global search.

A.3.1 Large Risky Positions

Lemma 1 leads to the following corollary on the properties for large risky positions, showing that the properties of the expected utility for large risky positions are governed by A.

Corollary 6. *(Expected utility for large risky positions.)*

1. *When $A > \underline{A}$, the expected utility $U(\phi)$ satisfies $U(\pm\infty) \to -\infty$, $U'(+\infty) < 0$, $U'(-\infty) > 0$, and $U''(\pm\infty) < 0$.*

2. *When $A < \underline{A}$, $U(\phi) \to +\infty$ for at least one of $\phi \to +\infty$ or $\phi \to -\infty$. If the infinite utility occurs at $\phi \to +\infty$ ($\phi \to -\infty$), then $U'(\phi) > 0$ ($U'(\phi) < 0$); in either case, $U''(\phi) > 0$.*

A.3.2 Small Risky Positions

Lemma 3. *(Expected utility for small risky positions.)*

1. *When $\theta > W_0 R_f$, the expected utility U is convex over the interval $\phi \in [\lambda, 0]$, and*

   (a) *U is increasing over the interval $\phi \in [\lambda, 0]$ for $\mu - r_f \geq \gamma\sigma^2$;*

   (b) *U is decreasing at $\phi = \lambda$ and increasing at $\phi = 0$ for $0 < \mu - r_f < \gamma\sigma^2$;*

   (c) *U is decreasing over the interval $\phi \in [\lambda, 0]$ for $\mu - r_f < 0$.*

2. *When $\theta < W_0 R_f$, U is concave over the interval $\phi \in [0, \lambda]$, and*

   (a) *U is increasing over the interval $\phi \in [0, \lambda]$ for $\mu - r_f \geq \gamma\sigma^2$;*

   (b) *U is decreasing at $\phi = \lambda$ and increasing at $\phi = 0$ for $0 < \mu - r_f < \gamma\sigma^2$;*

   (c) *U is decreasing over the interval $\phi \in [0, \lambda]$ for $\mu - r_f < 0$.*

3. *When $\theta = W_0 R_f$, $U = 0$ at $\phi = 0$ is a local maximum.*

Proof. When $\Lambda \equiv W_0 R_f - \theta < 0$,

$$U = \begin{cases} -A \int_0^{R_f - \frac{\Lambda}{x}} f(R_T) \frac{[-x(R_T - R_f) - \Lambda]^{1-\gamma}}{1-\gamma} dR_T + \int_{R_f - \frac{\Lambda}{x}}^{\infty} f(R_T) \frac{[x(R_T - R_f) + \Lambda]^{1-\gamma}}{1-\gamma} dR_T, & \text{if } x > 0; \\ -A \int_0^{\infty} f(R_T) \frac{[-x(R_T - R_f) - \Lambda]^{1-\gamma}}{1-\gamma} dR_T, & \text{if } \frac{\Lambda}{R_f} < x \leq 0; \\ \int_0^{R_f - \frac{\Lambda}{x}} f(R_T) \frac{[x(R_T - R_f) + \Lambda]^{1-\gamma}}{1-\gamma} dR_T - A \int_{R_f - \frac{\Lambda}{x}}^{\infty} f(R_T) \frac{[-x(R_T - R_f) - \Lambda]^{1-\gamma}}{1-\gamma} dR_T, & \text{if } x \leq \frac{\Lambda}{R_f}, \end{cases} \tag{A.1}$$

where $f(R_T) = \frac{1}{R_T\sigma\sqrt{2\pi T}}e^{-\frac{[\ln R_T-(\mu-\sigma^2/2)T]^2}{2\sigma^2 T}}$ is the density function of R_T. When $\frac{\Lambda}{R_f} < x \leq 0$, the gain domain ($W_T > \theta$) does not take effect, and $\frac{\partial U}{\partial x} = A\int_0^\infty f(R_T)[-x(R_T - R_f) - \Lambda]^{-\gamma}(R_T - R_f)dR_T$. Thus,

$$\frac{\partial U}{\partial x} = \begin{cases} A\int_0^\infty f(R_T)(-\Lambda)^{-\gamma}(R_T - R_f)dR_T = A(-\Lambda)^{-\gamma}R_f[e^{(\mu-r_f)T} - 1], & \text{if } x \uparrow 0; \\ A\int_0^\infty f(R_T)(-xR_T)^{-\gamma}(R_T - R_f)dR_T = A(-\frac{\Lambda}{R_f})^{-\gamma}R_f e^{-\gamma(\mu+\frac{1-\gamma}{2}\sigma^2)T}[e^{(\mu-r_f)T} - e^{\gamma\sigma^2 T}], & \text{if } x \downarrow \frac{\Lambda}{R_f}. \end{cases}$$

One can show that U is continuous and differentiable at $x = 0$ and $x = \frac{\Lambda}{R_f}$. In this interval $\frac{\Lambda}{R_f} < x \leq 0$, we have $\frac{\partial^2 U}{\partial x^2} = \gamma A\int_0^\infty f(R_T)[-x(R_T - R_f) - \Lambda]^{-\gamma-1}(R_T - R_f)^2 dR_T > 0$.

When $\Lambda > 0$,

$$U = \begin{cases} -A\int_0^{R_f-\frac{\Lambda}{x}} f(R_T)\frac{[-x(R_T-R_f)-\Lambda]^{1-\gamma}}{1-\gamma}dR_T + \int_{R_f-\frac{\Lambda}{x}}^\infty f(R_T)\frac{[x(R_T-R_f)+\Lambda]^{1-\gamma}}{1-\gamma}dR_T, & \text{if } x \geq \frac{\Lambda}{R_f}; \\ \int_0^\infty f(R_T)\frac{[x(R_T-R_f)+\Lambda]^{1-\gamma}}{1-\gamma}dR_T, & \text{if } 0 \leq x < \frac{\Lambda}{R_f}; \\ \int_0^{R_f-\frac{\Lambda}{x}} f(R_T)\frac{[x(R_T-R_f)+\Lambda]^{1-\gamma}}{1-\gamma}dR_T - A\int_{R_f-\frac{\Lambda}{x}}^\infty f(R_T)\frac{[-x(R_T-R_f)-\Lambda]^{1-\gamma}}{1-\gamma}dR_T, & \text{if } x < 0. \end{cases} \tag{A.2}$$

When $0 \leq x < \frac{\Lambda}{R_f}$, the loss domain ($W_T < \theta$) does not take effect, and U satisfies

$$\frac{\partial U}{\partial x} = \begin{cases} \int_0^\infty f(R_T)\Lambda^{-\gamma}(R_T - R_f)dR_T = \Lambda^{-\gamma}R_f[e^{(\mu-r_f)T} - 1], & \text{if } x \downarrow 0; \\ \int_0^\infty f(R_T)(xR_T)^{-\gamma}(R_T - R_f)dR_T = (\frac{\Lambda}{R_f})^{-\gamma}R_f e^{-\gamma(\mu+\frac{1-\gamma}{2}\sigma^2)T}[e^{(\mu-r_f)T} - e^{\gamma\sigma^2 T}], & \text{if } x \uparrow \frac{\Lambda}{R_f}. \end{cases}$$

and $\frac{\partial^2 U}{\partial x^2} < 0$ and is continuous and differentiable at $x = 0$ and $x = \frac{\Lambda}{R_f}$.

When $\Lambda = 0$,

$$U = \begin{cases} \big[-A\int_0^{R_f} f(R_T)(R_f - R_T)^{1-\gamma}dR_T + \int_{R_f}^\infty f(R_T)(R_T - R_f)^{1-\gamma}dR_T\big]\frac{x^{1-\gamma}}{1-\gamma}, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ \big[\int_0^{R_f} f(R_T)(R_f - R_T)^{1-\gamma}dR_T - A\int_{R_f}^\infty f(R_T)(R_T - R_f)^{1-\gamma}dR_T\big]\frac{(-x)^{1-\gamma}}{1-\gamma}, & \text{if } x < 0. \end{cases}$$

Under the boundedness conditions, $U < 0$ if $x \neq 0$. □

Lemmas 2-3 and Corollary 6 lead to the range of the optimal portfolio weights in Proposition 1.

Now we prove the sign of the value function. When $\Lambda = 0$, the global maximum of U is zero, as shown above. It follows from (A.1) and (A.2) that $\frac{\partial U}{\partial \Lambda} > 0$, where $\Lambda \equiv W_0 R_f - \theta$. Therefore, when $\Lambda < 0$, the expected utility is always smaller than the case of $\Lambda = 0$, independent of the portfolio weights. In addition, the expected utility is less than or equal to zero in the case of $\Lambda = 0$. Therefore, when $\Lambda < 0$, the expected utility and the value function are always negative. When $\Lambda > 0$, the expected utility can be negative. However, $U = \frac{\Lambda^{1-\gamma}}{1-\gamma}$ is positive if the agent holds only the riskless asset. Thus, the global maximum of the expected utility must be higher than or equal to this value, and must be positive.

A.4 Proof of Corollary 1

The portfolio wealth W_T is given by

$$W_T - \theta = xR_T + \big[(W_0 - x)R_f - \theta\big], \tag{A.3}$$

where the riskless return $R_f > 0$ is positive, and with lognormal distribution, the gross return of the risky asset satisfies $R_T \in (0, +\infty)$. The expected HARA utility is $-\infty$ if $W_T - \theta < 0$. Equation (A.3) shows that to have nonnegative $W_T - \theta$, both x and $[(W_0 - x)R_f - \theta]$ are nonnegative, which leads to either $W_0 R_f > \theta$ and $0 \le x \le W_0 - \frac{\theta}{R_f}$, or $W_0 R_f = \theta$ and $x = 0$.

Assume $W_0 R_f > \theta$. Define the portfolio weight of the risky asset as $\phi_{hara} = x/W_0$, which satisfies $0 \le \phi_{hara} \le \lambda$. The derivative of the expected utility is given by

$$\frac{\partial U_{hara}}{\partial \phi_{hara}} = W_0^{1-\gamma}\mathbb{E}\Big[\Big(R_f + \phi_{hara}(R_T - R_f) - \frac{\theta}{W_0}\Big)^{-\gamma}(R_T - R_f)\Big].$$

At $\phi_{hara} = 0$, it equals $W_0^{1-\gamma}(R_f - \frac{\theta}{W_0})^{-\gamma}\mathbb{E}[R_T - R_f]$, which has the same sign as the risk premium $\mathbb{E}[R_T - R_f] = e^{-r_f T}[e^{(\mu - r_f)T} - 1]$. At $\phi_{hara} = \lambda$, it equals $W_0^{1-\gamma}\lambda^{-\gamma}\mathbb{E}[R_T^{-\gamma}(R_T - R_f)] = W_0^{1-\gamma}\lambda^{-\gamma}e^{[r_f - \gamma\mu - \gamma(1-\gamma)\sigma^2/2]T}[e^{(\mu - r_f)T} - e^{\gamma\sigma^2 T}]$. In addition, $\frac{\partial^2 U}{\partial \phi_{hara}^2} = -\gamma W_0^{1-\gamma}\mathbb{E}[(R_f + \phi_{hara}(R_T - R_f) - \frac{\theta}{W_0})^{-\gamma-1}(R_T - R_f)^2] < 0$; thus, U is concave over $\phi_{hara} \in [0, 1]$.

When $\mu - r_f \ge \gamma\sigma^2$, U is increasing in $\phi_{hara} \in [0, \lambda]$, and its global maximum is at $\phi^*_{hara} = \lambda$. When $0 < \mu - r_f < \gamma\sigma^2$, U has the global maximum in $\phi_{hara} \in (0, \lambda)$. When $\mu - r_f \le 0$, U is decreasing in $\phi_{hara} \in [0, \lambda]$, and its global maximum is at $\phi^*_{hara} = 0$.

A.5 Proof of Corollary 3

The optimal portfolio weights follow from the FOCs for the expected utility function in (A.1) and (A.2), and the uniqueness follows from the monotonicity of $\frac{\partial U}{\partial \phi}$.

A.6 Proof of Proposition 2

The end of period wealth W_T can be written as

$$W_T - \theta = W_0 R_f - \theta + W_0 \phi (R_T - R_f), \tag{A.4}$$

where ϕ is the portfolio weight under the utility with a reference point θ. Define $\phi = \lambda \phi^\circ$. This is a one-to-one correspondence as long as $\lambda \neq 0$. We rewrite (A.4) in term of ϕ°:

$$W_T - \theta = \lambda W_0 \big[R_f + \phi^\circ (R_T - R_f) \big]. \tag{A.5}$$

The optimization problem can be written as

$$\max_{\phi} \mathbb{E}\Big[\hat{u}\Big(W_0 \big[R_f + \phi(R_T - R_f) \big] - \theta \Big) \Big] = \max_{\phi^\circ} \mathbb{E}\Big[\hat{u}\Big(\lambda W_0 \big[R_f + \phi^\circ (R_T - R_f) \big] \Big) \Big], \tag{A.6}$$

where $\hat{u}(w) = \begin{cases} \frac{1}{1-\gamma}(w)^{1-\gamma} & \text{if } w \geq 0; \\ -A\frac{1}{1-\gamma}(-w)^{1-\gamma} & \text{if } w < 0. \end{cases}$ Thus, the optimization problem with initial wealth W_0 and a reference point θ is equivalent to one with initial wealth λW_0 and a reference point 0. When $\lambda < 0$, the utility function with a reference point 0 is homogenous of degree $1-\gamma$ in its argument, and (A.6) becomes

$$\max_{\phi} \mathbb{E}\Big[\hat{u}\Big(W_0 \big[R_f + \phi(R_T - R_f) \big] - \theta \Big) \Big] = (-\lambda)^{1-\gamma} \max_{\phi^\circ} \mathbb{E}\Big[\hat{u}\Big(- W_0 \big[R_f + \phi^\circ (R_T - R_f) \big] \Big) \Big].$$

It shows that the expected utility with a reference point 0 and initial wealth $-W_0$ has its global maximum at $\phi^{\circ *} = \lambda^{-1} \phi^*$, where ϕ^* is the optimal portfolio weight for the original optimization problem with reference point θ and initial wealth W_0. The relationship between the optimal wealth under the two utility functions follows from (A.5).

When $\lambda > 0$, (A.6) becomes

$$\max_{\phi} \mathbb{E}\Big[\hat{u}\Big(W_0 \big[R_f + \phi(R_T - R_f) \big] - \theta \Big) \Big] = \lambda^{1-\gamma} \max_{\phi^\circ} \mathbb{E}\Big[\hat{u}\Big(W_0 \big[R_f + \phi^\circ (R_T - R_f) \big] \Big) \Big].$$

When $\lambda = 0$, the expected utility always equals 0 as shown in (A.6).

A.7 $\mathcal{C} = \mathcal{P}$ at $\xi = 0$

$$\mathbb{E}\Big[\Big(\frac{R_T}{R_f} - 1 \Big)^{1-\gamma} \mathbf{1}_{\{\frac{R_T}{R_f} - 1 \geq 0\}} \Big] = \mathbb{E}\Big[\Big(e^{(\mu - \frac{\sigma^2}{2} - r_f)T + \sigma\sqrt{T}\epsilon} - 1 \Big)^{1-\gamma} \mathbf{1}_{\{(\mu - \frac{\sigma^2}{2} - r_f)T + \sigma\sqrt{T}\epsilon \geq 0\}} \Big]. \tag{A.7}$$

If $\Delta \equiv \mu - r_f - \frac{\gamma\sigma^2}{2} = 0$, then (A.7) becomes

$$\begin{aligned}&\mathbb{E}\Big[\Big(e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\epsilon}-1\Big)^{1-\gamma}\mathbf{1}_{\{-\frac{1-\gamma}{2}\sigma\sqrt{T}+\epsilon\geq 0\}}\Big]\\ =&\mathbb{E}\Big[e^{-\frac{\sigma^2}{2}(1-\gamma)^2T+\sigma(1-\gamma)\sqrt{T}\epsilon}\Big(1-e^{\frac{\sigma^2}{2}(1-\gamma)T-\sigma\sqrt{T}\epsilon}\Big)^{1-\gamma}\mathbf{1}_{\{-\frac{1-\gamma}{2}\sigma\sqrt{T}+\epsilon\geq 0\}}\Big].\end{aligned} \tag{A.8}$$

Consider a new measure $\tilde{\mathbb{P}}$ defined by the Radon-Nikodym derivative: $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{\sigma^2}{2}(1-\gamma)^2T+\sigma(1-\gamma)\sqrt{T}\epsilon}$. Under $\tilde{\mathbb{P}}$, $\tilde{\epsilon} = \epsilon - \sigma(1-\gamma)\sqrt{T}$ is a standard normal random variable. Then (A.7) becomes

$$\begin{aligned}&\mathbb{E}\Big[\Big(\frac{R_T}{R_f}-1\Big)^{1-\gamma}\mathbf{1}_{\{\frac{R_T}{R_f}-1\geq 0\}}\Big] = \tilde{\mathbb{E}}\Big[\Big(1-e^{-\frac{\sigma^2}{2}(1-\gamma)T-\sigma\sqrt{T}\tilde{\epsilon}}\Big)^{1-\gamma}\mathbf{1}_{\{\frac{1-\gamma}{2}\sigma\sqrt{T}+\tilde{\epsilon}\geq 0\}}\Big]\\ =&\tilde{\mathbb{E}}\Big[\Big(1-e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\tilde{\epsilon}}\Big)^{1-\gamma}\mathbf{1}_{\{\frac{1-\gamma}{2}\sigma\sqrt{T}-\tilde{\epsilon}\geq 0\}}\Big] = \tilde{\mathbb{E}}\Big[\Big(1-e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\tilde{\epsilon}}\Big)^{1-\gamma}\mathbf{1}_{\{-\frac{1-\gamma}{2}\sigma\sqrt{T}+\tilde{\epsilon}<0\}}\Big]\\ =&\mathbb{E}\Big[\Big(1-e^{-\frac{\sigma^2}{2}(1-\gamma)T+\sigma\sqrt{T}\epsilon}\Big)^{1-\gamma}\mathbf{1}_{\{-\frac{1-\gamma}{2}\sigma\sqrt{T}+\epsilon<0\}}\Big] = \mathbb{E}\Big[\Big(1-\frac{R_T}{R_f}\Big)^{1-\gamma}\mathbf{1}_{\{1-\frac{R_T}{R_f}>0\}}\Big].\end{aligned}$$

B Effects of the Reference Point in the HARA Utility

We also have similar results of reference adjustment for the HARA utility.

Lemma 4. *The optimal portfolio weight and the optimal wealth for HARA utility with $\theta \leq W_0 R_f$ satisfies*

$$\phi^*_{hara} = \lambda\phi^{\circ *}_{hara}, \qquad W^*_T = \theta + \lambda W^{\circ *}_T, \tag{B.1}$$

where $\phi^{\circ *}_{hara}$ and $W^{\circ *}_T$ are the optimal portfolio weight and the optimal wealth for CRRA utility (with $\theta = 0$).

Proof. The proof is a special case of Appendix A.6. □

Lemma 4 shows that the optimal portfolio weight under HARA utility has a one-to-one correspondence with that under CRRA utility, and hence its properties can be understood from the optimal portfolio under CRRA utility that is widely studied in the literature.

Lemma 4 shows that a HARA agent with $\theta > 0$ trades less aggressively than the corresponding CRRA agent, consistent with Section 5.1.1.

C Nonpositive Initial Wealth

The loss aversion utility function proposed in Tversky and Kahneman (1992) generally allows nonpositive initial wealth. In this section, we study the case $W_0 \leq 0$.

First, we consider the case $W_0 < 0$. We assume the reference point is positive $\theta > 0$, which is the economic relevant case. In this case, $\lambda = 1 - \frac{\theta}{W_0 R_f} > 0$. The following proposition shows that the results with negative initial wealth can be understood from the case with positive initial wealth as studied above.

Proposition 3. *For the optimization problem under loss aversion with negative initial wealth $W_0 < 0$, the optimal portfolio weight and the optimal wealth satisfy*

$$\phi^* = \frac{\lambda}{\hat{\lambda}} \hat{\phi}^*, \qquad W_T^* - \theta = -\frac{\lambda}{\hat{\lambda}} (\hat{W}_T^* - \hat{\theta}), \tag{C.1}$$

where $\hat{\phi}^*$ and $\hat{W}_T^*$ are the optimal portfolio weight and the optimal wealth for the optimization problem with positive initial wealth $\hat{W}_0 = -W_0 > 0$ and reference point $\hat{\theta}$ that satisfies $\hat{\lambda} = 1 - \frac{\hat{\theta}}{\hat{W}_0 R_f} < 0$.

Proof. When $W_0 < 0$, it follows that $\lambda > 0$. Define $x = \lambda x^\circ$. We rewrite (A.6) as

$$\begin{aligned} \max_x \mathbb{E}\Big[u\Big(W_0 R_f + x(R_T - R_f) - \theta\Big)\Big] &= \max_{x^\circ} \mathbb{E}\Big[u\Big(\lambda\big[W_0 R_f + x^\circ(R_T - R_f)\big]\Big)\Big] \\ &= \lambda^{1-\gamma} \max_{x^\circ} \mathbb{E}\Big[u\Big(W_0 R_f + x^\circ(R_T - R_f)\Big)\Big]. \end{aligned}$$

It shows that the expected utility with a reference point 0 and initial wealth W_0 has its global maximum at $x^{\circ *} = \lambda^{-1} x^*$. Thus, $\phi^* = \lambda \phi^{\circ *}$, where the optimal portfolio weights $\phi^* \equiv x^*/W_0$ and $\phi^{\circ *} \equiv x^{\circ *}/W_0$, for $W_0 \neq 0$, and the optimal wealth satisfies $W_T^* = \theta + \lambda W_T^{\circ *}$.

In addition, Appendix A.6 shows that the expected utility with a reference point 0 and initial wealth $W_0 < 0$ has its global maximum at $\phi^{\circ *} = \hat{\lambda}^{-1} \hat{\phi}^*$, where $\hat{\phi}^*$ is the optimal portfolio weight for the optimization problem with reference point $\hat{\theta}$ and initial wealth $\hat{W}_0 = -W_0 > 0$ that satisfy $\hat{\lambda} = 1 - \frac{\hat{\theta}}{\hat{W}_0 R_f} < 0$. The optimal wealth follows $\hat{W}_T^* = \theta - \hat{\lambda} W_T^{\circ *}$. □

Proposition 3 shows that the results with negative initial wealth is symmetric to those with positive initial wealth. When initial wealth is negative, the reference adjustment factor λ is always positive. The optimal portfolio weight can be mapped to the case with positive initial wealth $\hat{W}_0 > 0$ and a negative reference adjustment factor $\hat{\lambda} < 0$.

With zero initial wealth $W_0 = 0$, we have $\theta > W_0 R_f$. The result is the same as the case $\lambda < 0$ as in Proposition 1, except that in this case we use dollar demand x^*, as generally studied in the Appendix, since the portfolio weight is not well-defined.

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